Lecture 5: Quiz (Simulation of the Posterior Distribution)

Instructor: Thibault Randrianarisoa

Questions

Question 1 What is a primary limitation of the Inverse Transform method for simulating a random variable?

a) It requires calculating the exact ratio of the target density to a proposal density to determine the acceptance probability.

b) It requires the ability to invert the cumulative distribution function (CDF), which is not always explicitly possible.

c) It has an acceptance rate that depends strictly on a bounding constant m.

d) It only works for random variables that are defined on discrete, finite state spaces.

Question 2 In Rejection Sampling, suppose we use a proposal density g to simulate a target density f, and we find a constant $m \geq 1$ such that $f(y) \leq mg(y)$ for all y. What is the expected number of trials needed to obtain one accepted simulation?

a) $1/m$

b) m

c) $1 - (1/m)$

d) m^2

Question 3 Why is Importance Sampling often preferred over Standard Monte Carlo for estimating the probability of rare events?

a) It guarantees an estimator variance of exactly zero regardless of the chosen proposal density.

b) It breaks the dependence on the dimension of the integral without requiring the generation of any random points.

c) It allows the use of a proposal density that places more probability mass in the specific regions where the integrand is large, thereby drastically reducing variance.

d) It automatically simulates directly from the difficult target distribution without needing an intermediate proposal distribution.

Question 4 How does the Gibbs sampler conceptually relate to the Metropolis-Hastings (MH) algorithm according to the lecture?

a) The Gibbs sampler is a completely independent methodology that does not rely on Markov Chains, unlike MH.

b) The Gibbs sampler randomly rejects proposals more frequently than a standard MH algorithm to ensure faster convergence to the stationary distribution.

c) The Gibbs sampler is an extension of MH that is strictly used when the exact acceptance probability cannot be computed analytically.

d) The Gibbs sampler is a special case of MH where the proposal distributions are the full conditional distributions, resulting in an acceptance probability of exactly 1.

Question 5 What is the key distinction between an irreducible transition kernel and an ergodic transition kernel on a finite state space?

a) For an irreducible kernel, the required time t to reach state y from state x can depend on x and y, whereas an ergodic kernel requires a single time t such that $P^t(x, y) > 0$ for all pairs x, y simultaneously.

b) Irreducibility guarantees that the chain's distribution converges to the stationary distribution as $t \to \infty$, while ergodicity only guarantees that the stationary distribution is unique.

c) An ergodic kernel always satisfies the detailed balance condition, whereas an irreducible kernel does not.

d) Ergodicity implies that the chain can reach any state from any other state in exactly 1 step ($t = 1$), whereas irreducibility allows for $t > 1$.

Solutions

1. Correct Answer: b) It requires the ability to invert the cumulative distribution function (CDF), which is not always explicitly possible.
Reasoning: The Inverse Transform method relies on applying the generalized inverse of the CDF (F^{-1}) to a uniform random variable. For distributions like the Gaussian, this inverse lacks an explicit closed form, making the method difficult to use directly.

2. Correct Answer: b) m
Reasoning: The number of trials required to accept a sample follows a geometric distribution with a success probability of $1/m$. Therefore, the expected value (mean) of trials until the first success is exactly m.

3. Correct Answer: c) It allows the use of a proposal density that places more probability mass in the specific regions where the integrand is large, thereby drastically reducing variance.
Reasoning: Standard Monte Carlo requires massive sample sizes to observe rare events because most random draws fall outside the region of interest. Importance sampling solves this by intentionally drawing from a proposal distribution focused on the rare region and correcting the bias with weighting ratios.

4. Correct Answer: d) The Gibbs sampler is a special case of MH where the proposal distributions are the full conditional distributions, resulting in an acceptance probability of exactly 1.
Reasoning: By using the exact conditional distributions as the proposal mechanism for moving between states, the Metropolis-Hastings acceptance ratio simplifies perfectly to 1, meaning that every proposed move in a Gibbs sampler is automatically accepted.

8. Correct Answer: a) For an irreducible kernel, the required time t to reach state y from state x can depend on x and y, whereas an ergodic kernel requires a single time t such that $P^t(x, y) > 0$ for all pairs x, y simultaneously.
Reasoning: The definition of irreducibility states that for all $x, y \in \Omega$, there exists a time $t \in \mathbb{N}$ such that $P^t(x, y) > 0$, meaning the time t depends on the specific starting and ending states x and y. In contrast, ergodicity requires a stronger condition: there exists a specific time $t \in \mathbb{N}$ such that for all $x, y \in \Omega$, $P^t(x, y) > 0$. Additionally, irreducibility guarantees a unique stationary distribution that charges all states, but it is ergodicity that guarantees the chain actually converges to that stationary measure.