STAD91: Quiz 3 Practice (Decision Theory)

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Questions

Question 1 Consider a Beta-Binomial model where $X|\theta \sim \text{Bin}(n, \theta)$ and the prior is $\theta \sim \text{Beta}(4, 5)$. Under the squared error loss (L^2), what is the posterior risk $\rho(\pi, T|X)$?

a) $\frac{X+4}{n+9}$

b) $\frac{(X+4)(n-X+5)}{(n+9)^2(n+10)}$

c) $\frac{(X+1)(n-X+1)}{(n+2)^2(n+3)}$

d) $\frac{1}{n+10}$

Question 2 Let $X_1, \dots, X_n$ be an i.i.d. sample from a Gaussian distribution $\mathcal{N}(\theta, 1)$. With a Gaussian prior $\pi = \mathcal{N}(0, 1)$ on θ, what is the Bayes Estimator under the *Absolute Loss* function?

a) $\bar{X}_n$

b) $\frac{n}{n+1}\bar{X}_n$

c) $|\bar{X}_n|$

d) Zero

Question 3 For the L^2-loss in the Gaussian model $\mathcal{N}(\theta, 1)$, consider the estimator $T = \bar{X}_n + \beta$ where $\beta \neq 0$ is a fixed non-zero constant. Is this estimator admissible?

a) Yes, because it is of the form $\alpha\bar{X}_n + \beta$ described in the slides.

b) Yes, because it has constant risk.

c) No, it is inadmissible because it is strictly dominated by $\bar{X}_n$.

d) No, because it is a Bayes estimator.

Question 4 Which of the following properties applies to the sample mean $\bar{X}_n$ in the Gaussian model $\mathcal{N}(\theta, 1)$ under quadratic loss? (Select the one that is **FALSE**)

a) It is Admissible.

b) It is Minimax.

c) It has Constant Risk.

d) It is a Bayes Estimator for a Gaussian prior.

Solutions

1. Correct Answer: b) $\frac{(X+4)(n-X+5)}{(n+9)^2(n+10)}$
Reasoning: The prior is Beta(4, 5). Given x successes in n trials, the posterior updates to Beta(α', β') where $\alpha' = 4 + x$ and $\beta' = 5 + n - x$. For L^2 loss, the Posterior Risk is the posterior variance. The variance of a Beta(α', β') distribution is $\frac{\alpha'\beta'}{(\alpha'+\beta')^2(\alpha'+\beta'+1)}$. Note that $\alpha' + \beta' = n + 9$. Substituting these values yields the result.

2. Correct Answer: b) The posterior median, which equals $\frac{n}{n+1}\bar{X}_n$
Reasoning: For absolute loss, the Bayes estimator is the Posterior Median. Since the posterior distribution is Gaussian $\mathcal{N}(\frac{n\bar{X}_n}{n+1}, \frac{1}{n+1})$, it is symmetric, meaning the median equals the mean.

3. Correct Answer: c) No, it is inadmissible...
Reasoning: The risk of T is $E[(\bar{X}_n + \beta - \theta)^2] = 1/n + \beta^2$. The risk of the sample mean $\bar{X}_n$ is $1/n$. Since $\beta \neq 0$, T has strictly higher risk for all θ, making it inadmissible. Note that estimators of the form $\alpha\bar{X}_n + \beta$ are admissible if $\alpha \in (0, 1)$, but here $\alpha = 1$.

4. Correct Answer: d) It is a Bayes Estimator for a proper prior.
Reasoning: While $\bar{X}_n$ is admissible, minimax, and has constant risk, it is *not* a Bayes estimator for any proper (integrable) prior. It corresponds to the limit of Bayes estimators as the prior variance goes to infinity (improper flat prior).