PRACTICE MIDTERM EXAM

STAD91 WINTER 2026
University of Toronto Scarborough

Exam duration: 2H
No calculators will be allowed during the midterm exam.
A formula sheet with the usual families of distributions will be provided during the actual midterm.

Read the following instructions carefully:

1. Exam is closed book and internet. You can use an optional handwritten aid sheet (A4 double-sided).
2. If a question asks you to do some calculations, you must show your work for full credit.
3. Conceptual questions do not require long answers.
4. You will write your answers to each question in the space provided on the exam sheet.
5. After solving each question, you should write your answers immediately. Do not wait last minute to write them all at once.
6. Do not share the exam with anyone or in any platform!
7. Lastly, enjoy the problems!!!

1. Exponential Models and Bayes Estimators (30 pts)

Let $n \geqslant 3$. Consider the following Bayesian framework:

$$\theta \sim \Pi = \mathcal{E}(1)$$
$$\mathbf{X}|\theta \sim P_\theta^{\otimes n} = \mathcal{E}(\theta)^{\otimes n}$$

1. Determine the posterior distribution.
2. Consider the loss function $\ell(\theta, t) = e^\theta (t-\theta)^2$. Give the Bayes estimator $T^* = T^*(\mathbf{X})$ for Π and this loss function.
3. Show that, under $P_\theta^{\otimes n}$, the variable $n\overline{X}_n$ follows a Gamma distribution and specify its parameters. Deduce $\mathbb{E}_\theta\left[\frac{1}{n\overline{X}_n}\right]$ and $\mathbb{E}_\theta\left[\frac{1}{(n\overline{X}_n)^2}\right]$.
4. Show that the point risk $R(\theta, T^*) = \mathbb{E}_\theta[\ell(\theta, T^*)]$ can be written in the form:
$$R(\theta, T^*) = \frac{an+b}{(n-1)(n-2)}\theta^2 e^\theta$$
where a and b are two positive constants to be specified.
5. What is the Bayes risk $R_B(\Pi)$? Deduce the minimax risk R_M.
6. We wish to test
$$H_0 : \theta = 1 \quad \text{against} \quad H_1 : \theta = 2.$$
To do this, we now consider the prior $\Pi = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_2$ and the balanced loss function. Show that the Bayes test can be expressed in the form $\varphi^*(\mathbf{X}) = \mathbb{1}_{\{\overline{X}_n \leqslant c\}}$, for a certain constant $c \in \mathbb{R}$ that you will determine.

2. Gaussian Model and Empirical Bayes (25 pts)

Let $\lambda > 0$ and $n \geqslant 5$ be an integer. Consider the following Bayesian model:

$$\theta \sim \Pi = \mathcal{E}(\lambda)$$
$$\mathbf{X} = (X_1, \ldots, X_n)|\theta \sim \mathcal{N}\left(0, \frac{1}{\theta}\right)^{\otimes n}$$

1. Determine the posterior distribution $\Pi[\cdot|\mathbf{X}]$. We denote $S_{\mathbf{X}}^2 = \sum_{i=1}^n X_i^2$.

2. Give a Bayes estimator for the quadratic loss.

3. Consider the estimator $\widehat{\theta}_n(\mathbf{X}) = \frac{n}{S_{\mathbf{X}}^2}$. The objective of this question is to calculate its Bayes risk for Π and the quadratic loss.

   (a) Let $\theta > 0$. Given $\theta = \theta$, what distribution does $\theta S_{\mathbf{X}}^2$ follow?

   (b) Deduce, for all $\theta > 0$, $\mathbb{E}_\theta[\frac{1}{S_{\mathbf{X}}^2}]$ and $\mathbb{E}_\theta[\frac{1}{S_{\mathbf{X}}^4}]$.

   (c) Deduce that for all $\theta > 0$, the point risk is $R(\theta, \widehat{\theta}_n(\mathbf{X})) = \frac{2(n+4)}{(n-2)(n-4)}\theta^2$.

4. Use the Empirical Bayes method to calibrate λ. Show that for all $\lambda > 0$, the marginal density $f_\lambda(\mathbf{X})$ satisfies:

$$f_\lambda(\mathbf{X}) \propto \frac{\lambda}{(\lambda + \frac{S_{\mathbf{X}}^2}{2})^{\frac{n}{2}+1}}$$

   and determine the marginal maximum likelihood estimator of λ.

3. Poisson Distribution and improper priors (25 pts)

Let X be a random variable following a Poisson distribution $\mathcal{P}(\theta)$ with $\theta \in \mathbb{R}_+^*$, and $x_1, \ldots, x_n$ a sample from this distribution.

1. Determine the Jeffreys prior measure $\pi^J(\theta)$.
2. Evaluate, based on the existence of posterior distributions, which prior is more suitable between $\pi_0 \propto 1/\theta$ and π^J.
3. Let $\pi_\alpha(\theta) \propto \theta^{-\alpha}$ with $\alpha \in \mathbb{R}^+$.
   (a) Show that the posterior distribution $p(\theta|x_1, \ldots, x_n)$ is a Gamma distribution and specify its shape parameter A and rate parameter B.
   (b) Write down the integral definition for the posterior predictive mass function $P(X_{new} = k|x_1, \ldots, x_n)$ using the posterior density you found in the previous step.
   (c) Compute the integral to find the closed-form expression for $P(X_{new} = k|x_1, \ldots, x_n)$ in terms of Gamma functions. Identify the name of this known distribution.
   (d) Provide the expressions for its expectation and variance, and state their conditions for existence.

4. Bayesian Linear Regression (20 pts)

Consider a simple linear regression model where we assume the intercept is zero and the noise variance σ^2 is known. We observe data $\mathcal{D} = \{(x_i, y_i)\}_{i=1}^n$. The model is:

$$y_i = \beta x_i + \epsilon_i, \quad \epsilon_i \sim N(0, \sigma^2)$$

We place a Gaussian prior on the slope parameter: $\beta \sim N(\mu_0, \tau_0^2)$.

1. Write down the likelihood $p(\mathbf{y}|\mathbf{x}, \beta)$. Express it as a single multivariate normal distribution.
2. Derive the posterior distribution $p(\beta|\mathbf{x}, \mathbf{y})$. Show that $\beta|\mathbf{y} \sim N(\mu_n, \tau_n^2)$ and find expressions for the posterior precision $(1/\tau_n^2)$ and the posterior mean (μ_n).
3. Interpret the expression for the posterior precision. How does the "sample information" and the "prior information" combine to determine our certainty about β?
4. Suppose we use a "flat" prior by letting $\tau_0^2 \to \infty$. Show that the posterior mean μ_n converges to the ordinary least-square estimator $\hat{\beta}_{OLS} = \frac{\sum x_i y_i}{\sum x_i^2}$.
5. What is the distribution of the posterior predictive $p(y^*|x^*, \mathbf{x}, \mathbf{y})$ for a new observation y^* at a new given point x^*?