# STAD91: Uniqueness of Bayes Estimators

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Lecture 3 Addendum

## Uniqueness of Bayes Estimators

In Lecture 3, we discussed the admissibility of unique Bayes estimators. The following theorem provides the rigorous conditions under which a Bayes estimator is unique.

**Theorem 1** (Uniqueness of Bayes Estimators)**.** *Let $f$ be the marginal density of $X$ given the prior $\pi$, defined by:*

$$f(x) = \int_{\Theta} p_\theta(x) d\pi(\theta).$$

*Suppose the loss function $l(\theta, \cdot)$ is strictly convex for every $\theta$. If:*

1. *The Bayes risk is finite, i.e., $R_B(\pi) < \infty$, and*
2. *The model is dominated by the marginal in the sense that:*

$$\int_E f(x)dx = 0 \implies \int_E p_\theta(x)dx = 0, \quad \forall \theta \in \Theta,$$

*then the Bayes estimator $T$ is unique (up to equivalence).*

*Proof.* Suppose that $T_1$ and $T_2$ are two Bayes estimators with respect to the prior $\pi$. By definition, they both achieve the minimum Bayes risk:

$$R_B(\pi, T_1) = R_B(\pi, T_2) = \inf_T R_B(\pi, T) < \infty.$$

Consider the estimator $T' = \frac{1}{2}T_1 + \frac{1}{2}T_2$. Due to the strict convexity of the loss function $l(\theta, \cdot)$, for any $x$ such that $T_1(x) \neq T_2(x)$, we have strict inequality:

$$l\left(\theta, \frac{1}{2}T_1(x) + \frac{1}{2}T_2(x)\right) < \frac{1}{2}l(\theta, T_1(x)) + \frac{1}{2}l(\theta, T_2(x)).$$

Taking the expectation with respect to the joint distribution of $(X, \theta)$, we obtain the Bayes risk of $T'$:

$$\begin{aligned} R_B(\pi, T') &= \int_\Theta \int_E l(\theta, T'(x)) dP_\theta(x) d\pi(\theta) \\ &< \int_\Theta \int_E \left(\frac{1}{2}l(\theta, T_1(x)) + \frac{1}{2}l(\theta, T_2(x))\right) dP_\theta(x) d\pi(\theta) \\ &= \frac{1}{2}R_B(\pi, T_1) + \frac{1}{2}R_B(\pi, T_2) \\ &= R_B(\pi). \end{aligned}$$

The strict inequality holds unless $T_1(X) = T_2(X)$ almost everywhere with respect to the marginal distribution of $X$. If the inequality were strict, we would have $R_B(\pi, T') < R_B(\pi)$, which contradicts the fact that $T_1$ and $T_2$ minimize the risk.

Therefore, we must have $T_1(x) = T_2(x)$ for almost all $x$. That is, if $A = \{x : T_1(x) \neq T_2(x)\}$, then $\int_A f(x)dx = 0$. Finally, by condition 2, it implies $\int_A p_\theta(x)dx = 0$ for all $\theta$. Thus, $T_1$ and $T_2$ are equal almost everywhere with respect to $P_\theta$ for all $\theta$, meaning $T_1$ and $T_2$ are equivalent. □