

# STAD91 Bayesian Statistics

## Lecture 1: Organization & Review

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UTSC

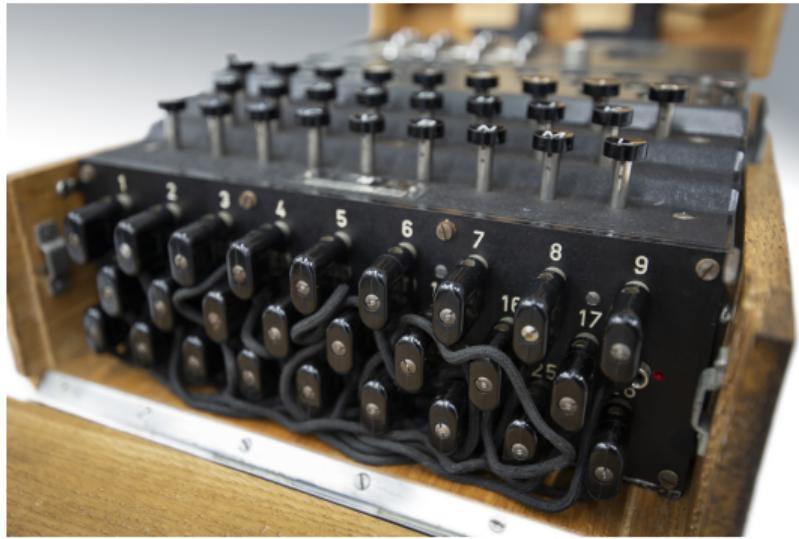
January 8, 2026



# Overview of the lecture

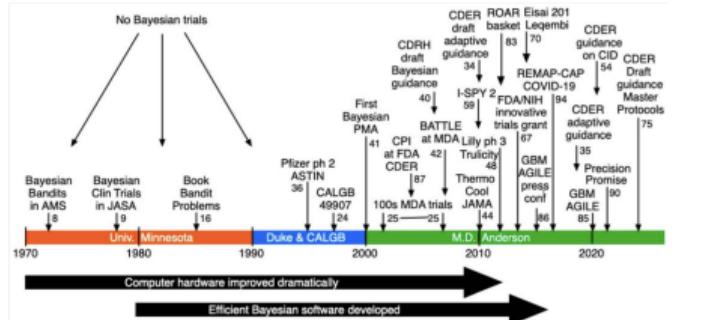
- First part: Organization of the course
  - Motivation
  - Course logistics
  - Assessment
- Second part: Background from Probability and Statistics
  - Measure theory and random variables
  - Statistical models and estimators
  - Model selection
  - High-dimensional / nonparametric models

# World War II



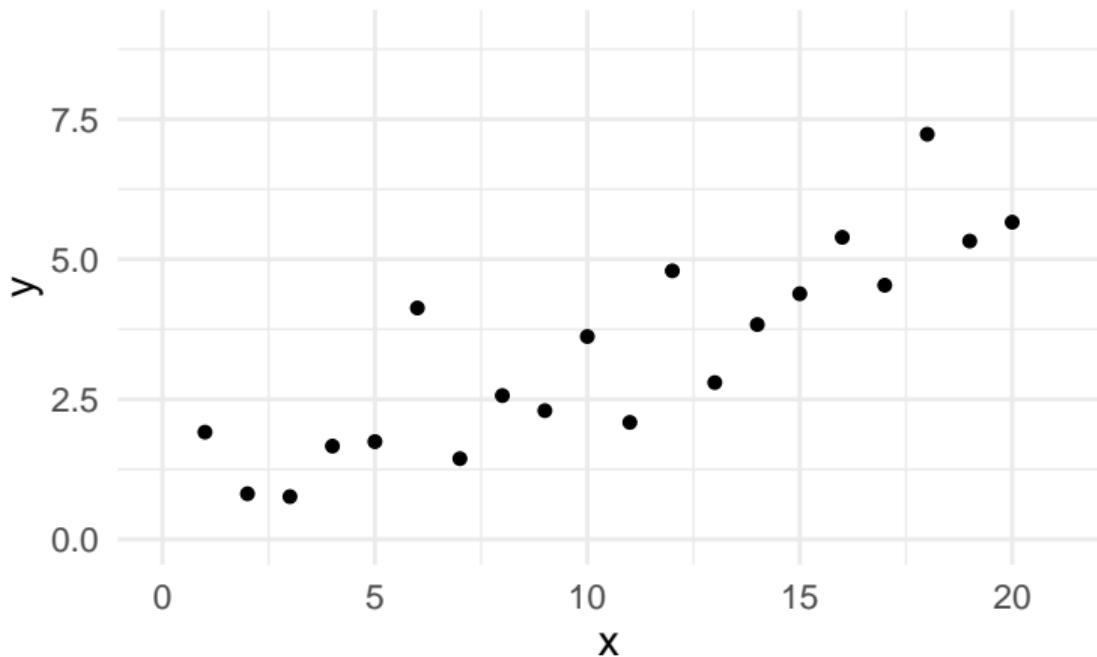
# Search for plane wreck





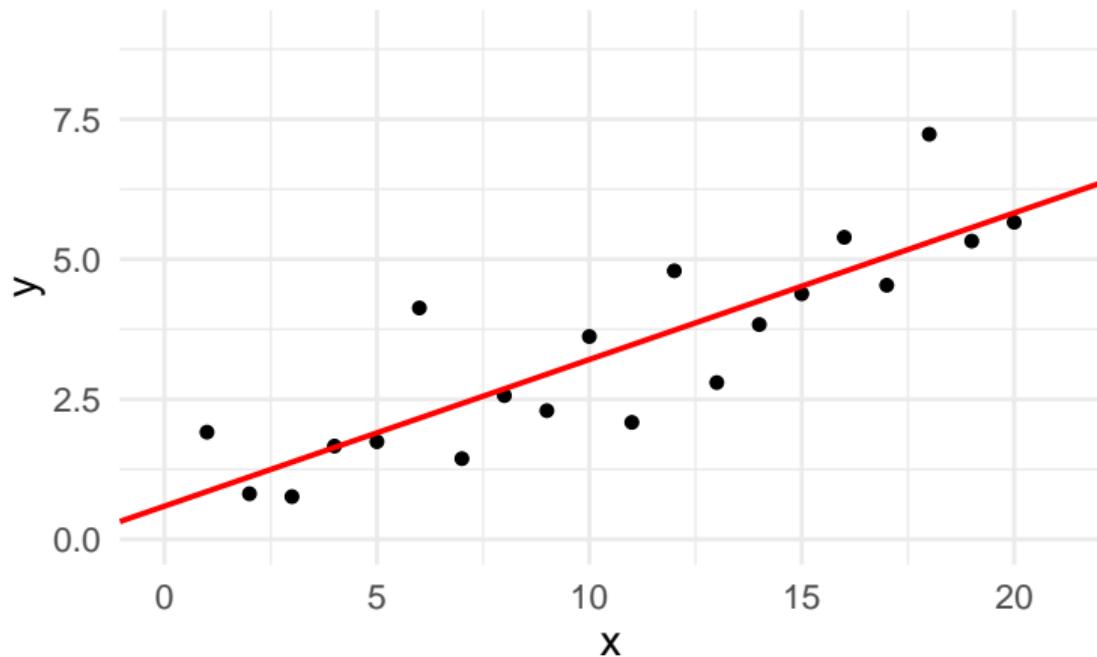
## Uncertainty in modeling

### Data



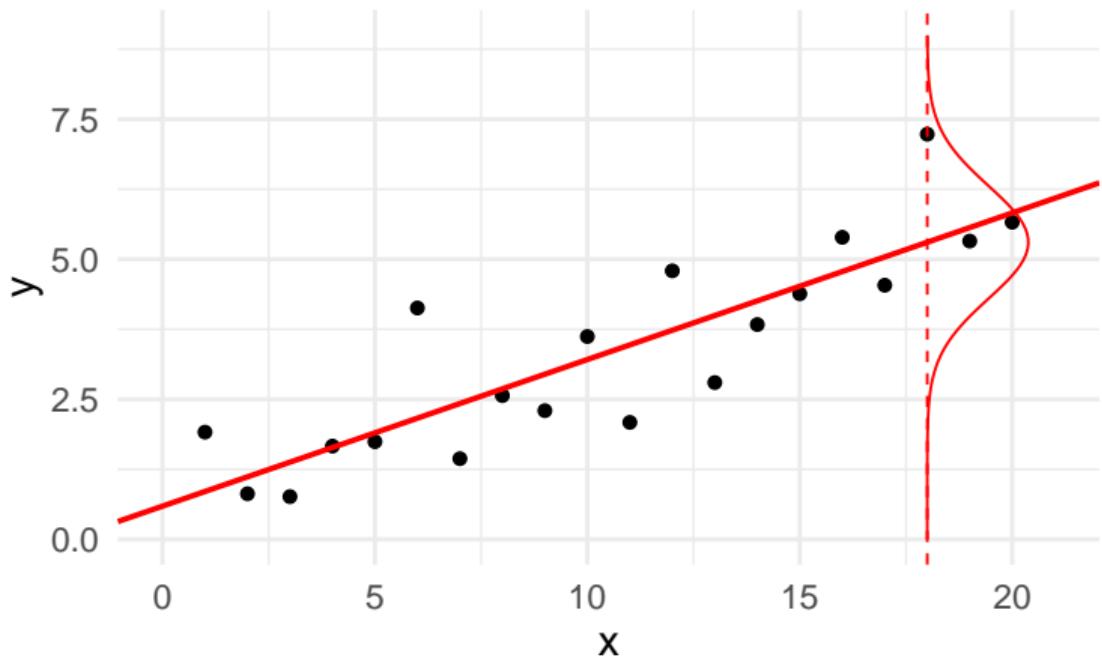
## Uncertainty in modeling

### Posterior mean



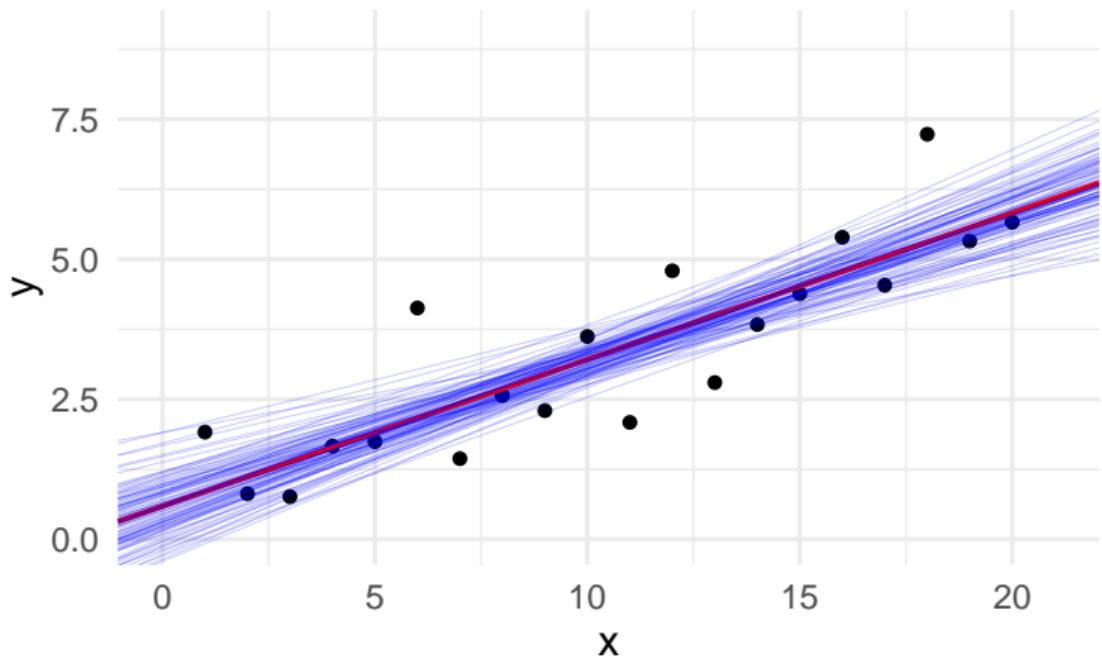
## Uncertainty in modeling

Predictive distribution given posterior mean



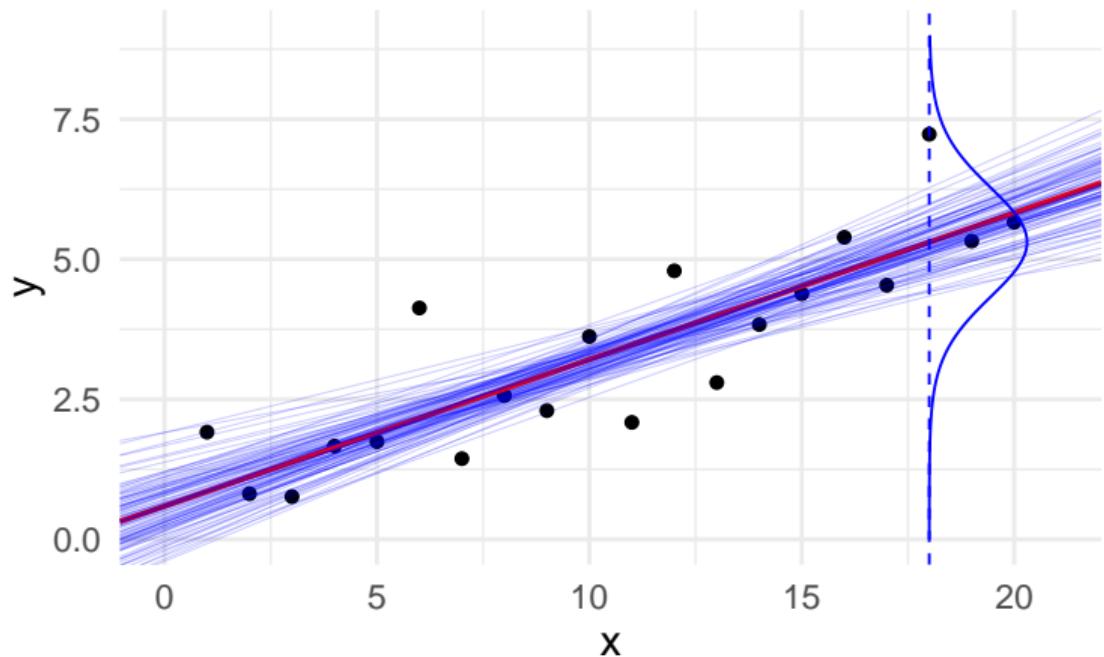
## Uncertainty in modeling

### Posterior draws



## Uncertainty in modeling

### Posterior draws and predictive distribution



# Uncertainty and probabilistic modeling

- Representing uncertainty with **probabilities** + Updating **uncertainty**
- Two types of uncertainty: aleatoric and epistemic
- Aleatoric uncertainty due to randomness
  - we are not able to obtain observations which could reduce this uncertainty
- Epistemic uncertainty due to lack of knowledge
  - we are able to obtain observations which can reduce this uncertainty
  - two observers may have different epistemic uncertainty

Better modelling and quantification of uncertainty

- better science
- better informed decision making  
in companies, government, and NGOs

# Bayesian probability theory

expert information, previous experiments,...

## Bayesian probability theory

expert information, previous experiments,...



mathematical model

+

uncertainty with probabilities

## Bayesian probability theory

expert information, previous experiments,...

data



mathematical model

+

uncertainty with probabilities

## Bayesian probability theory

expert information, previous experiments,...



mathematical model

+

uncertainty with probabilities

+

Bayesian probability theory

$$p(\theta|y) = \frac{p(y|\theta)p(\theta)}{p(y)}$$

$$p(\tilde{y}|y) = \int p(\tilde{y}|\theta)p(\theta|y)d\theta$$



data

# Bayesian probability theory

expert information, previous experiments,...



mathematical model

+

uncertainty with probabilities

+

Bayesian probability theory

$$p(\theta|y) = \frac{p(y|\theta)p(\theta)}{p(y)}$$

$$p(\tilde{y}|y) = \int p(\tilde{y}|\theta)p(\theta|y)d\theta$$



data



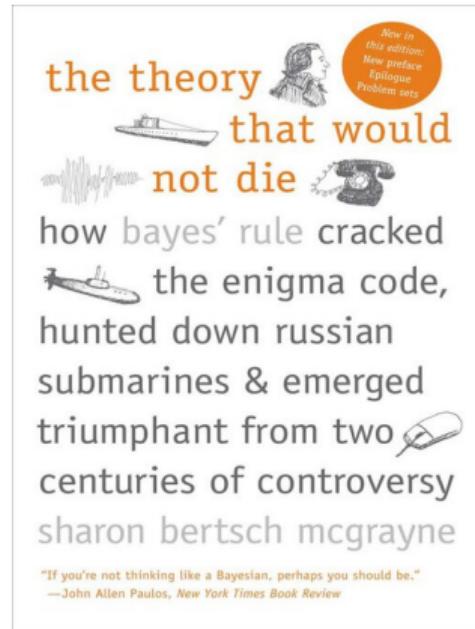
updated uncertainty

# Bayesian probability theory

- Based on Bayesian probability theory
  - uncertainty is presented with probabilities
  - probabilities are updated based on new information
- Thomas Bayes (170?–1761)
  - English nonconformist, Presbyterian minister, (amateur) mathematician
  - considered the problem of *inverse probability*
    - significant part of the Bayesian theory
- Bayes did not invent all, but was first to solve problem of inverse probability in special case
- Laplace generalized the initial methods and applied it to scientific problems (e.g., astronomy)
- Modern Bayesian theory with rigorous proofs developed in 20th century

## Bayesian probability theory

A nice book about history: Sharon Bertsch McGrayne, *The Theory That Would Not Die*, 2012.



## Term Bayesian used first time in mid 20th century

- Earlier there was just "probability theory"
  - concept of the probability was not strictly defined, although it was close to modern Bayesian interpretation
  - in the end of 19th century there were increasing demand for more strict definition of probability (mathematical and philosophical problem)
- In the beginning of 20th century frequentist view gained popularity
  - accepts definition of probabilities only through frequencies
  - does not accept inverse probability or use of prior
  - gained popularity due to apparent objectivity and "cook book" like reference books
- R. A. Fisher used in 1950 first time term "Bayesian" to emphasize the difference to general term "probability theory"
  - term became quickly popular, because alternative descriptions were longer
- The probabilistic programming revolution started in early 1990's

# Bayesian Statistics course

- Probability distributions as model building blocks
  - need to understand the math part (prereq.)
  - continuous vs discrete (prereq.)
  - observation model, likelihood, prior
  - constructing bigger models
- Computation
  - We need to be able to compute expectations

$$E_{\theta|y}[g(\theta)] = \int p(\theta|y)g(\theta)d\theta$$

- when analytic solutions are not available, computational approximations with finite number of function evaluations
- importance sampling, Monte Carlo, Markov chain Monte Carlo, variational Bayes
- **Not in this course:** Diagnostics

## Bayesian Statistics course

- *Bayesian inference* : process of statistical learning via Bayes' rule.

$$P(A|E) = \frac{P(E|A)P(A)}{P(E|A)P(A) + P(E|A^c)P(A^c)} = \frac{P(E|A)P(A)}{P(E)}$$

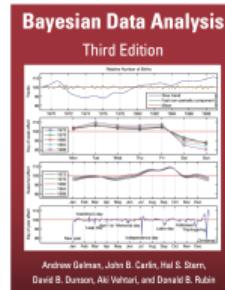
or

$$p(\theta|y) = \frac{p(\theta, y)}{p(y)} = \frac{p(\theta)p(y|\theta)}{p(y)}$$

- *Bayesian methods* are data analysis tools that are derived from the principles of Bayesian inference.
- Bayesian methods provide:
  - a rational method for updating beliefs in light of new information;
  - parameter estimates with good statistical properties;
  - predictions for missing data and forecasts of future data;
  - a computational framework for model estimation, selection and validation.

# Course Logistics

- Course administered through [Quercus](#)
  - Syllabus, Lecture Notes, HW Problems, Quizzes, etc
  - Also look at the [course webpage](#)
- Textbook: [Bayesian Data Analysis](#) by Gelman, Carlin, Stern, Dunson, Vehtari & Rubin.



- Communication
  - For course content questions use [Piazza](#) or OH
  - For personal issues use email (with [STAD91] in the subject) or office hours (TH 10am-1pm @ IA 4064)

## Assessment

Evaluation	Weight	Details
Weekly Quizzes (on Kahoot)	20%	<ul style="list-style-type: none"><li>• Best 8/10</li><li>• Cover previous week's material</li><li>• At the end of the lecture</li></ul>
Homework Assignments	10%	<ul style="list-style-type: none"><li>• Two Homework Assignments (5% each)</li><li>• Date: After the midterm, TBD</li><li>• Pen &amp; paper derivations + Coding (Python/Numpy or R)</li></ul>
Term Test	25%	<ul style="list-style-type: none"><li>• Covers first 5 weeks</li><li>• Tentative Date: Feb 26 (wk 10)</li></ul>
Final	45%	<ul style="list-style-type: none"><li>• Cumulative</li><li>• Final exam period</li></ul>

No make-up Quizzes/Term Tests; if you miss midterm you must **declare absence on ACORN** & mark will be replaced by final.

# Review of Key Concepts

Probability & inference topics you are expected to remember (quick recap)

## Absolute continuity

### Definition (absolute continuity)

Let  $P$  and  $\mu$  be two  $\sigma$ -finite measures on a measurable space  $(E, \mathcal{E})$ . We say that  $P$  is **absolutely continuous** with respect to  $\mu$ , and write  $P \ll \mu$ , if

$$\forall A \in \mathcal{E}, \quad \mu(A) = 0 \implies P(A) = 0.$$

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### Radon–Nikodym theorem

If  $P \ll \mu$ , then there exists a positive measurable function  $p$  such that for every  $A \in \mathcal{E}$ ,

$$P(A) = \int_A p(x) d\mu(x).$$

The function  $p$  is called the **Radon–Nikodym derivative** of  $P$  with respect to  $\mu$  and is denoted

$$p = \frac{dP}{d\mu}.$$

## Absolute continuity

### Notation

One may write, suggestively,

$$P(A) = \int_A dP(x) = \int_A \frac{dP(x)}{d\mu(x)} d\mu(x) = \int_A p(x) d\mu(x).$$

### Discrete distributions

- On  $E = \{0, 1\}$ , the Bernoulli( $\theta$ ) law has a density with respect to  $\mu = \delta_0 + \delta_1$ :

$$p(x) = (1 - \theta)\mathbf{1}_{\{x=0\}} + \theta\mathbf{1}_{\{x=1\}}.$$

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- On  $E = \{0, 1, \dots, n\}$ , the Binomial( $n, \theta$ ) law is absolutely continuous with respect to the counting measure  $\mu = \sum_{k=0}^n \delta_k$ , with density

$$p(k) = \binom{n}{k} \theta^k (1 - \theta)^{n-k}, \quad k = 0, 1, \dots, n.$$

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- On  $E = \mathbb{N}^*$ , the geometric law with parameter  $p$  is absolutely continuous with respect to the counting measure  $\mu = \sum_{k \geq 1} \delta_k$ , with density

$$p(k) = (1 - p)^{k-1} p, \quad k \geq 1.$$

## Continuous distributions

- The normal law  $\mathcal{N}(\mu, \sigma^2)$  has density with respect to Lebesgue measure on  $\mathbb{R}$ :

$$x \longmapsto \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}.$$

- The exponential law with rate  $\lambda > 0$  has density (with respect to Lebesgue on  $\mathbb{R}$ )

$$x \longmapsto \lambda e^{-\lambda x} \mathbf{1}_{\{x \geq 0\}}.$$

## Markov's inequality

Let  $X$  be a non-negative real random variable and  $a > 0$ . Then

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[X]}{a}.$$

In particular, for a real random variable  $X$  and  $p \in \mathbb{N}^*$ , since  $x \mapsto x^p$  is increasing on  $\mathbb{R}_+$ ,

$$\mathbb{P}(|X| \geq a) = \mathbb{P}(|X|^p \geq a^p) \leq a^{-p} \mathbb{E}[|X|^p].$$

## Chebyshev's inequality

Let  $X$  be a real random variable and  $a > 0$ . Then

$$\mathbb{P}(|X - \mathbb{E}[X]| \geq a) \leq \frac{\text{Var}(X)}{a^2}, \quad \text{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2].$$

## Hoeffding's inequality

Let  $X_1, \dots, X_n$  be independent random variables, denote  $\bar{X}_n = \frac{1}{n} \sum_{k=1}^n X_k$  and suppose  $a_i \leq X_i \leq b_i$  a.s.. Then for all  $\varepsilon \geq 0$ ,

$$\mathbb{P}(\bar{X}_n - \mathbb{E}[\bar{X}_n] \geq \varepsilon) \leq \exp\left(-\frac{2\varepsilon^2 n^2}{\sum_{i=1}^n (b_i - a_i)^2}\right).$$

### Example: Bernoulli sample mean

Let  $X_1, \dots, X_n$  be i.i.d. with  $\text{Bernoulli}(p)$  distribution and  $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$ . For every  $\varepsilon > 0$ ,

### Chebyshev's inequality:

$$\mathbb{P}(|\bar{X}_n - p| > \varepsilon) \leq \frac{\text{Var}(\bar{X}_n)}{\varepsilon^2} = \frac{p(1-p)}{n\varepsilon^2}.$$

Since  $0 \leq X_i \leq 1$ , we can improve this using **Hoeffding's inequality**:

$$\mathbb{P}(|\bar{X}_n - p| > \varepsilon) \leq 2 \exp(-2n\varepsilon^2).$$

Chebyshev gives a bound of order  $1/n$ , whereas Hoeffding yields an exponentially small bound in  $n$ .

# Gamma and Beta distributions

## Gamma distribution

For  $p, \lambda > 0$ , a random variable  $Z$  has  $\text{Gamma}(p, \lambda)$  distribution if it has density

$$f_Z(x) = \frac{\lambda^p}{\Gamma(p)} x^{p-1} e^{-\lambda x} \mathbf{1}_{\{x \geq 0\}}, \quad \Gamma(p) = \int_0^\infty z^{p-1} e^{-z} dz.$$

- $\mathbb{E}[Z] = \frac{p}{\lambda}$ ,  $\text{Var}(Z) = \frac{p}{\lambda^2}$ .
- Special case:  $\Gamma(1, \lambda) = \text{Exp}(\lambda)$ .

## Beta distribution

For  $a, b > 0$ , a random variable  $X$  has  $\text{Beta}(a, b)$  distribution if it has density

$$f_X(x) = \frac{1}{B(a, b)} x^{a-1} (1-x)^{b-1} \mathbf{1}_{\{0 \leq x \leq 1\}}, \quad B(a, b) = \int_0^1 z^{a-1} (1-z)^{b-1} dz = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

- $\mathbb{E}[X] = \frac{a}{a+b}$ ,  $\text{Var}(X) = \frac{ab}{(a+b)^2(a+b+1)}$ .
- Special case:  $\text{Beta}(1, 1) = \mathcal{U}[0, 1]$ .

## Gamma and Beta: main properties

### Additivity of Gamma

If  $Y \sim \Gamma(p, \lambda)$  and  $Z \sim \Gamma(q, \lambda)$  are independent, then

$$Y + Z \sim \Gamma(p + q, \lambda).$$

In particular, if  $E_1, \dots, E_n$  are i.i.d.  $\text{Exp}(\lambda)$ , then  $\sum_{i=1}^n E_i \sim \Gamma(n, \lambda)$ .

### Scaling of Gamma

If  $Y \sim \Gamma(p, \lambda)$  and  $t > 0$ , then  $tY \sim \Gamma(p, \frac{\lambda}{t})$ .

### Gamma–Beta connection

If  $X \sim \Gamma(a, \lambda)$  and  $Y \sim \Gamma(b, \lambda)$  are independent, then  $\frac{X}{X+Y} \sim \text{Beta}(a, b)$ .

As a special case, if  $E_1, E_2$  are i.i.d.  $\text{Exp}(\lambda)$ , then

$$\frac{E_1}{E_1 + E_2} \sim \mathcal{U}[0, 1].$$

# Dirichlet distribution: definition and properties

## Definition (Dirichlet distribution)

Let  $K \geq 2$  and  $\alpha_1, \dots, \alpha_K > 0$ . A random vector  $X = (X_1, \dots, X_K)$  has  $\text{Dirichlet}(\alpha_1, \dots, \alpha_K)$  distribution if  $X_i > 0$ ,  $\sum_{i=1}^K X_i = 1$ , and its density on the *simplex* is

$$f_X(x_1, \dots, x_K) = \frac{\Gamma(\sum_{i=1}^K \alpha_i)}{\prod_{i=1}^K \Gamma(\alpha_i)} \prod_{i=1}^K x_i^{\alpha_i-1}, \quad (x_1, \dots, x_K) \in S_K = \{x \in [0, 1]^K : \sum_i x_i = 1\}$$

## Key properties

- **Beta as a special case:** for  $K = 2$ ,  $\text{Dir}(\alpha_1, \alpha_2)$  is the  $\text{Beta}(\alpha_1, \alpha_2)$  distribution.
- **Marginals are Beta:** if  $X \sim \text{Dir}(\alpha_1, \dots, \alpha_K)$ , then

$$X_i \sim \text{Beta}\left(\alpha_i, \sum_{k=1}^K \alpha_k - \alpha_i\right), \quad \mathbb{E}[X_i] = \frac{\alpha_i}{\sum_{k=1}^K \alpha_k}$$

- **Gamma representation:** If  $Z_i \sim \Gamma(\alpha_i, \lambda)$  are independent and  $Z = \sum_{k=1}^K Z_k$ , then

$$\left(\frac{Z_1}{Z}, \dots, \frac{Z_K}{Z}\right) \sim \text{Dir}(\alpha_1, \dots, \alpha_K).$$

# Modes of convergence of random variables

## Convergence in probability

Let  $X_1, \dots, X_n, \dots$  and  $X$  be random variables taking values in  $\mathbb{R}^d$ , defined on the same probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . The sequence  $(X_n)$  *converges in probability* to  $X$ , written  $X_n \xrightarrow{\mathbb{P}} X$ , if

$$\forall \varepsilon > 0, \quad \mathbb{P}(\|X_n - X\| > \varepsilon) \xrightarrow{n \rightarrow \infty} 0.$$

## Convergence in $L^2$

In the same setting, we say that  $(X_n)$  *converges in  $L^2$*  to  $X$ , written  $X_n \xrightarrow{L^2} X$ , if

$$\mathbb{E} \left[ \|X_n - X\|^2 \right] \xrightarrow{n \rightarrow \infty} 0.$$

## Modes of convergence of random variables (II)

### Almost sure convergence

With the same notation, the sequence  $(X_n)$  *converges almost surely* to  $X$ , written  $X_n \xrightarrow{\text{a.s.}} X$  if

$$\mathbb{P} \left( \left\{ \omega \in \Omega : X_n(\omega) \xrightarrow[n \rightarrow \infty]{} X(\omega) \right\} \right) = 1.$$

### Proposition

We have the implications

$$X_n \xrightarrow{\text{a.s.}} X \implies X_n \xrightarrow{\mathbb{P}} X,$$

and

$$X_n \xrightarrow{L^2} X \implies X_n \xrightarrow{\mathbb{P}} X.$$

## Convergence in distribution

### Convergence in distribution / in law

Let  $(X_n)_{n \geq 1}$  and  $X$  be random variables with values in  $\mathbb{R}^d$ . We say that  $X_n$  *converges in distribution* (or in law) to  $X$ , written  $X_n \xrightarrow{\mathcal{L}} X$ , if for every **bounded continuous** function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ ,

$$\mathbb{E}[f(X_n)] \xrightarrow{n \rightarrow \infty} \mathbb{E}[f(X)].$$

Similarly, we say that  $(X_n)$  converges in distribution to a probability measure  $P$  on  $\mathbb{R}^d$  if

$$\mathbb{E}[f(X_n)] \xrightarrow{n \rightarrow \infty} \mathbb{E}[f(X)]$$

for  $X \sim P$  and every bounded continuous function  $f$ .

## Central Limit Theorem in $\mathbb{R}^d$

### Multivariate Central Limit Theorem

Let  $(X_n)$  be a sequence of i.i.d. random variables with values in  $\mathbb{R}^d$ , such that  $\mathbb{E} [\|X_1\|^2] < \infty$ .

Let

$$\mu = \mathbb{E}[X_1], \quad \Sigma = \mathbb{E} [(X_1 - \mathbb{E}[X_1])(X_1 - \mathbb{E}[X_1])^T].$$

Then

$$\sqrt{n} (\bar{X}_n - \mu) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \Sigma).$$

## Continuous mapping theorem, Slutsky's lemma

### Continuous mapping theorem

Let  $X_n, X$  be random variables taking values in  $\mathbb{R}^d$  and  $g : \mathbb{R}^d \rightarrow \mathbb{R}^k$  a continuous function.

- If  $X_n \xrightarrow{\mathcal{L}} X$ , then  $g(X_n) \xrightarrow{\mathcal{L}} g(X)$ .
- If  $X_n \xrightarrow{\mathbb{P}} X$ , then  $g(X_n) \xrightarrow{\mathbb{P}} g(X)$ .
- If  $X_n \xrightarrow{\text{a.s.}} X$ , then  $g(X_n) \xrightarrow{\text{a.s.}} g(X)$ .

### Slutsky's lemma

Let  $(X_n)$  and  $(Y_n)$  be sequences of real-valued random variables,  $X$  a real-valued random variable, and  $a \in \mathbb{R}$ .

$$X_n \xrightarrow{\mathcal{L}} X \quad \text{and} \quad Y_n \xrightarrow{\mathbb{P}} a \implies (X_n, Y_n) \xrightarrow{\mathcal{L}} (X, a).$$

### Remark

For a **constant**  $a$ , we have

$$Z_n \xrightarrow{\mathcal{L}} a \iff Z_n \xrightarrow{\mathbb{P}} a.$$

## Statistical experiment and model

A *statistical experiment* consists of

- a random variable  $X$  defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with values in a measurable space  $(E, \mathcal{E})$ ;
- a family of probability measures on  $(E, \mathcal{E})$ , called a *statistical model*,

$$\mathcal{P} = \{P_\theta : \theta \in \Theta\},$$

where  $\Theta$  is the *parameter space*.

In the *frequentist approach* one assumes that the law of  $X$  belongs to the model:

$$\exists \theta_0 \in \Theta, \quad X \sim P_{\theta_0}.$$

Statistical inference aims at learning about  $\theta_0$  from an observation of  $X$ .

## Sample model

In practice  $X$  is often an  $n$ -tuple of random variables

$$X = (X_1, \dots, X_n)$$

Then the sample space and the model depend on  $n$ .

### Example: $n$ -sample model

When  $X = (X_1, \dots, X_n)$ , one often works with the  **$n$ -sample model**

$$\mathcal{P}_n = \{P_\theta^{\otimes n} : \theta \in \Theta\},$$

where

$$P_\theta^{\otimes n} = \underbrace{P_\theta \otimes \cdots \otimes P_\theta}_{n \text{ times}}.$$

This corresponds to assuming that  $X_1, \dots, X_n$  are i.i.d. with common distribution  $P_\theta$ .

# Identifiability and dominated models

## Identifiable model

A statistical model  $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$  is *identifiable* if for all  $\theta, \theta' \in \Theta$ ,

$$P_\theta = P_{\theta'} \implies \theta = \theta'.$$

Equivalently, the mapping  $\theta \mapsto P_\theta$  is injective. This guarantees that each distribution in the model corresponds to a unique parameter value.

## Dominated model

The model  $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$  is *dominated* if there exists a  $\sigma$ -finite measure  $\mu$  on  $E$  such that, for all  $\theta \in \Theta$ ,  $P_\theta \ll \mu$ . Then every  $P_\theta$  admits a density  $p_\theta$  with respect to  $\mu$ :

$$dP_\theta(x) = p_\theta(x) d\mu(x).$$

In what follows we often work with dominated, *parametric models* with  $\Theta \subset \mathbb{R}^d$ .

## Example 1: Bernoulli model

Consider  $E = \{0, 1\}$  and parameter space  $\Theta = (0, 1)$ . For  $\theta \in \Theta$  let

$$P_\theta(X = 1) = \theta, \quad P_\theta(X = 0) = 1 - \theta.$$

The model is

$$\mathcal{P} = \{P_\theta : \theta \in (0, 1)\}.$$

- This is a dominated model with respect to the counting measure on  $\{0, 1\}$ ; the density is

$$p_\theta(x) = (1 - \theta)\mathbf{1}_{\{0\}}(x) + \theta\mathbf{1}_{\{1\}}(x).$$

- The model is identifiable:  $P_\theta = P_{\theta'}$  implies  $\theta = P_\theta(X = 1) = P_{\theta'}(X = 1) = \theta'$ .

## Example 2: Gaussian model with unknown mean

Let  $E = \mathbb{R}$ ,  $\Theta = \mathbb{R}$  and fix  $\sigma^2 > 0$ . For  $\theta \in \Theta$  define  $P_\theta$  as the normal law

$$P_\theta = \mathcal{N}(\theta, \sigma^2).$$

The model is

$$\mathcal{P} = \{\mathcal{N}(\theta, \sigma^2) : \theta \in \mathbb{R}\}.$$

- This model is dominated by Lebesgue measure  $\lambda$  on  $\mathbb{R}$  with density

$$p_\theta(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\theta)^2}{2\sigma^2}\right).$$

- It is identifiable: equality of the densities (or distributions) for all  $x$  forces the means to be equal.

# Estimators in a statistical experiment

## Estimator

Consider the statistical experiment  $(X, \mathcal{P})$ . An *estimator* of the parameter  $\theta$  is a measurable function

$$\hat{\theta} = \hat{\theta}(X)$$

with values in the parameter space  $\Theta$  (more precisely,  $\hat{\theta}$  is measurable from  $(E, \mathcal{E})$  to  $(\Theta, \mathcal{B}(\Theta))$ , where  $\mathcal{B}(\Theta)$  is the Borel  $\sigma$ -algebra).

## Sequence of experiments

In practice we often have a sequence of experiments  $(X^{(n)}, \mathcal{P}_n)$ ,  $n = 1, 2, \dots$

This leads to a sequence of estimators  $(\hat{\theta}_n)$ .

## Likelihood and maximum likelihood estimator

Assume a dominated model with respect to a measure  $\mu$ : for each  $\theta \in \Theta$ ,

$$dP_\theta(x) = \mathbf{p}_\theta(x) d\mu(x).$$

Let  $X = (X_1, \dots, X_n) \sim P_\theta^{\otimes n}$ . The **joint density** of  $X$  is

$$p_\theta^{\otimes n}(x_1, \dots, x_n) = \prod_{i=1}^n p_\theta(x_i).$$

Viewed as a function of  $\theta$  for the observed data  $X$ , this is the *likelihood function*

$$L_\theta(X) = \prod_{i=1}^n p_\theta(X_i).$$

Often we work instead with the *log-likelihood*

$$\ell_\theta(X) = \log L_\theta(X) = \sum_{i=1}^n \log \mathbf{p}_\theta(X_i).$$

# Maximum likelihood estimator (MLE)

## Definition (MLE)

In a dominated model, a *maximum likelihood estimator (MLE)* is, when it exists, a value  $\hat{\theta}(X) \in \Theta$  such that

$$\hat{\theta}(X) \in \arg \max_{\theta \in \Theta} L_{\theta}(X), \quad \text{or equivalently} \quad \hat{\theta}(X) \in \arg \max_{\theta \in \Theta} \ell_{\theta}(X).$$

**Example / exercise (Bernoulli model).** In the Bernoulli model  $\mathcal{P} = \{\mathcal{B}(\theta)^{\otimes n} : \theta \in [0, 1]\}$ , show that the MLE of  $\theta$  is unique and given by the empirical mean

$$\hat{\theta}(X) = \bar{X}_n.$$

# Maximum likelihood estimator (MLE)

# Consistency and asymptotic normality

## Consistency

Consider a sequence of experiments  $(X^{(n)}, \mathcal{P}_n)$  with

$$\mathcal{P}_n = \{P_\theta^{\otimes n} : \theta \in \Theta\}.$$

A sequence of estimators  $(\hat{\theta}_n)$  is *consistent* if, for every  $\theta \in \Theta$ , when  $X^{(n)} \sim P_\theta^{\otimes n}$ ,

$$\hat{\theta}_n(X^{(n)}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \theta.$$

## Asymptotic normality

In the same setting,  $(\hat{\theta}_n)$  is *asymptotically normal* if for each  $\theta \in \Theta$  there exists a symmetric positive semi-definite matrix  $\Sigma_\theta$  such that, when  $X^{(n)} \sim P_\theta^{\otimes n}$ ,

$$\sqrt{n}(\hat{\theta}_n(X^{(n)}) - \theta) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, \Sigma_\theta).$$

## Consistency and asymptotic normality

**Exercise.** Show that if  $(\hat{\theta}_n)$  is asymptotically normal, then it is consistent.

## Quadratic risk of an estimator

### Definition (Quadratic risk)

Let  $(X, \mathcal{P})$  be a statistical experiment with  $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$  and let  $\hat{\theta}$  be an estimator. The *quadratic risk* of  $\hat{\theta}$  at  $\theta$  is

$$R(\theta, \hat{\theta}) = \mathbb{E}_\theta \left[ \left\| \hat{\theta}(X) - \theta \right\|^2 \right] = \int_E \left\| \hat{\theta}(x) - \theta \right\|^2 dP_\theta(x).$$

### Example: Scalar parameter case

When  $\Theta \subset \mathbb{R}$ , the quadratic risk reduces to

$$R(\theta, \hat{\theta}) = \mathbb{E}_\theta \left[ (\hat{\theta}(X) - \theta)^2 \right] = \int_E (\hat{\theta}(x) - \theta)^2 dP_\theta(x).$$

A "good" estimator typically has small quadratic risk, but remember that  $R(\theta, \hat{\theta})$  is a function of  $\theta$  and may be small for some parameter values and large for others.

## Bias–variance decomposition

### Proposition (Bias–variance decomposition)

Let  $(X, \mathcal{P})$  be a statistical experiment with  $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$  and let  $\hat{\theta}$  be an estimator. For every  $\theta \in \Theta$ , if  $X \sim P_\theta$ ,

$$R(\theta, \hat{\theta}) = \mathbb{E}_\theta \left[ \left\| \hat{\theta}(X) - \mathbb{E}_\theta[\hat{\theta}(X)] \right\|^2 \right] + \left\| \mathbb{E}_\theta[\hat{\theta}(X)] - \theta \right\|^2.$$

The function

$$\theta \longmapsto \mathbb{E}_\theta[\hat{\theta}(X)] - \theta$$

is called the *bias* of  $\hat{\theta}$ .

### Scalar parameter case

If  $\Theta \subset \mathbb{R}$ , then

$$R(\theta, \hat{\theta}) = \text{Var}_\theta(\hat{\theta}(X)) + (\mathbb{E}_\theta[\hat{\theta}(X)] - \theta)^2.$$

## Example: Bernoulli model and empirical mean

### Setting

Let  $(X, \mathcal{P})$  with  $\mathcal{P} = \{\mathcal{B}(\theta)^{\otimes n} : \theta \in [0, 1]\}$ , where  $X = (X_1, \dots, X_n)$  and  $X_i$  are i.i.d.  $\text{Bernoulli}(\theta)$ .

A natural estimator of  $\theta$  is the empirical mean

$$\hat{\theta}_n(X) = \bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i.$$

- By the (strong) law of large numbers,  $\hat{\theta}_n(X) \rightarrow \theta$  almost surely, hence  $\hat{\theta}_n$  is consistent.
- By the central limit theorem,  $\sqrt{n}(\hat{\theta}_n(X) - \theta) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \theta(1 - \theta))$ , so  $\hat{\theta}_n$  is asymptotically normal.
- Since  $\mathbb{E}_\theta[\hat{\theta}_n(X)] = \theta$ , the estimator is unbiased and

$$R(\theta, \hat{\theta}_n) = \mathbb{E}_\theta \left[ (\hat{\theta}_n(X) - \theta)^2 \right] = \text{Var}_\theta(\hat{\theta}_n(X)) = \frac{\theta(1 - \theta)}{n}.$$

## Risk and probability of large error

For any estimator  $\hat{\theta}$  and any  $\varepsilon > 0$ , the quadratic risk controls the probability of a large error:

$$\mathbb{P}_\theta \left( |\hat{\theta}(X) - \theta| \geq \varepsilon \right) \leq \frac{\mathbb{E}_\theta \left[ (\hat{\theta}(X) - \theta)^2 \right]}{\varepsilon^2} = \frac{R(\theta, \hat{\theta})}{\varepsilon^2}.$$

This follows from Markov's (or Chebyshev's) inequality.

Thus, a small quadratic risk implies that  $\hat{\theta}(X)$  is close to  $\theta$  with high probability.

## Example: Gaussian mean, two estimators

### Setting

Let  $X_1, \dots, X_n$  be i.i.d.  $\mathcal{N}(\theta, 1)$  with  $\theta \in \mathbb{R}$ .

We compare two estimators:

- a constant estimator  $\tilde{\theta}_n = \theta_0$  for some fixed  $\theta_0 \in \mathbb{R}$ .

$$R(\theta, \tilde{\theta}_n) = \mathbb{E}_\theta[(\theta_0 - \theta)^2] = (\theta - \theta_0)^2.$$

This risk is zero at  $\theta = \theta_0$ , but positive elsewhere and does not decrease with  $n$ .

- the empirical mean  $\hat{\theta}_n(X) = \bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$ , for which  $\mathbb{E}_\theta[\hat{\theta}_n(X)] = \theta$  (unbiased) and

$$R(\theta, \hat{\theta}_n) = \text{Var}_\theta(\hat{\theta}_n(X)) = \frac{1}{n}.$$

The risk is independent of  $\theta$  and decreases at rate  $1/n$ .

## Consistency and asymptotic normality

**Exercise.** For  $X \sim \text{Bin}(n, \theta)$  and  $\hat{\theta} = X/n$ , show that  $R(\theta, \hat{\theta}) \leq 1/(4n)$  for all  $\theta \in [0, 1]$ .

## Exact confidence intervals and regions

Let  $\alpha > 0$ .

**Definition** (exact confidence interval / region)

- **Case**  $\Theta \subset \mathbb{R}$ . A (random) interval  $I(X) = [a(X), b(X)]$  is a *confidence interval of level (at least)  $1 - \alpha$*  if

$$\forall \theta \in \Theta, \quad \mathbb{P}_\theta(\theta \in I(X)) \geq 1 - \alpha.$$

- **Case**  $\Theta \subset \mathbb{R}^d$ . A random subset  $\mathcal{R}(X) \subset \Theta$  is a *confidence region of level (at least)  $1 - \alpha$*  if

$$\forall \theta \in \Theta, \quad \mathbb{P}_\theta(\theta \in \mathcal{R}(X)) \geq 1 - \alpha.$$

⚠ The interval  $I(X)$  cannot depend on the unknown parameter  $\theta$ ; it may only depend on known quantities (such as  $\alpha$ , the sample size  $n$ , and the data  $X$ ).

## Example: normal mean, exact confidence interval

- Gaussian model

We observe  $X = (X_1, \dots, X_n)$  i.i.d. with  $X_i \sim \mathcal{N}(\theta, 1)$ ,  $\theta \in \mathbb{R}$ . Let  $\hat{\theta}(X) = \bar{X}_n = n^{-1} \sum_{i=1}^n X_i$ . Then

$$\sqrt{n}(\bar{X}_n - \theta) \sim \mathcal{N}(0, 1).$$

## Example: normal mean, exact confidence interval

- Gaussian model

We observe  $X = (X_1, \dots, X_n)$  i.i.d. with  $X_i \sim \mathcal{N}(\theta, 1)$ ,  $\theta \in \mathbb{R}$ . Let  $\hat{\theta}(X) = \bar{X}_n = n^{-1} \sum_{i=1}^n X_i$ . Then

$$\sqrt{n}(\bar{X}_n - \theta) \sim \mathcal{N}(0, 1).$$

Denote by  $\Phi$  the c.d.f. of  $\mathcal{N}(0, 1)$  and set  $q_\alpha = \Phi^{-1}(1 - \alpha/2)$ , so that  $\mathbb{P}(|\mathcal{N}(0, 1)| > q_\alpha) = \alpha$ .

## Example: normal mean, exact confidence interval

- Gaussian model

We observe  $X = (X_1, \dots, X_n)$  i.i.d. with  $X_i \sim \mathcal{N}(\theta, 1)$ ,  $\theta \in \mathbb{R}$ . Let  $\hat{\theta}(X) = \bar{X}_n = n^{-1} \sum_{i=1}^n X_i$ . Then

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- Resulting confidence interval

We have

$$\mathbb{P}_\theta \left( \left| \sqrt{n}(\hat{\theta}(X) - \theta) \right| > q_\alpha \right) = \alpha.$$

Equivalently,

$$I(X) = \left[ \hat{\theta}(X) \pm \frac{q_\alpha}{\sqrt{n}} \right]$$

is an **exact** level  $1 - \alpha$  confidence interval for  $\theta$ .

# Asymptotic confidence intervals

Sometimes the finite-sample distribution of an estimator is unknown, but its limiting distribution as  $n \rightarrow \infty$  is known. This leads to asymptotic confidence intervals/regions.

## Definition (asymptotic confidence interval/region)

- **Case  $\Theta \subset \mathbb{R}$ .** A random interval  $I(X^{(n)})$  is an *asymptotic confidence interval of level (at least)  $1 - \alpha$*  if

$$\forall \theta \in \Theta, \quad \liminf_{n \rightarrow \infty} \mathbb{P}_\theta(\theta \in I(X^{(n)})) \geq 1 - \alpha.$$

- **Case  $\Theta \subset \mathbb{R}^d$ .** A random set  $\mathcal{R}(X^{(n)}) \subset \Theta$  is an *asymptotic confidence region of level (at least)  $1 - \alpha$*  if

$$\forall \theta \in \Theta, \quad \liminf_{n \rightarrow \infty} \mathbb{P}_\theta(\theta \in \mathcal{R}(X^{(n)})) \geq 1 - \alpha.$$

## General construction from an asymptotically normal estimator

### Proposition (asymptotic CI from asymptotic normality)

Assume  $\Theta \subset \mathbb{R}$  and let  $\hat{\theta}_n = \hat{\theta}_n(X)$  be an estimator such that

$$\sqrt{n} (\hat{\theta}_n - \theta) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, \sigma^2(\theta)),$$

where the function  $\theta \mapsto \sigma^2(\theta)$  is continuous.

Let  $q_\alpha > 0$  satisfy

$$\mathbb{P}(|\mathcal{N}(0, 1)| \leq q_\alpha) = 1 - \alpha \quad (\text{so } q_\alpha = \Phi^{-1}(1 - \alpha/2)).$$

Define

$$I(X) = \left[ \hat{\theta}_n(X) - \frac{q_\alpha \sigma(\hat{\theta}_n(X))}{\sqrt{n}}, \hat{\theta}_n(X) + \frac{q_\alpha \sigma(\hat{\theta}_n(X))}{\sqrt{n}} \right].$$

Then  $I(X)$  is an **asymptotic confidence interval of level exactly  $1 - \alpha$** .

# General construction from an asymptotically normal estimator

proof:

## Conditional distribution (discrete case)

### Definition (discrete conditional law)

Let  $X$  and  $Y$  be discrete random variables on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , with values respectively in sets  $E$  and  $F$ . For  $x \in E$  such that  $\mathbb{P}(X = x) > 0$ , the **conditional distribution of  $Y$  given  $X = x$** , denoted  $\mathcal{L}(Y | X = x)$ , is defined for all  $y \in F$  by

$$\mathbb{P}(Y = y | X = x) = \frac{\mathbb{P}(Y = y, X = x)}{\mathbb{P}(X = x)}.$$

This defines, for each fixed  $x$ , a probability distribution on  $F$ .

## Joint densities and marginals

Let

- $(E, \mathcal{E})$  and  $(F, \mathcal{F})$  be measurable spaces;
- $\alpha$  a  $\sigma$ -finite positive measure on  $(E, \mathcal{E})$ , and  $\beta$  a  $\sigma$ -finite positive measure on  $(F, \mathcal{F})$ ;
- $X$  an  $E$ -valued random variable and  $Y$  an  $F$ -valued random variable.

Assume the pair  $(X, Y)$  has a joint density  $h(x, y)$  with respect to  $\alpha \otimes \beta$ , i.e.

$$dP(x, y) = h(x, y) d\alpha(x) d\beta(y).$$

The **marginal law of  $X$**  is the (probability) density

$$f(x) = \int_F h(x, y) d\beta(y),$$

and the **marginal law of  $Y$**  is the (probability) density

$$g(y) = \int_E h(x, y) d\alpha(x).$$

## Conditional density (continuous case)

### Definition (conditional density for fixed $x$ )

Assume  $f(x) > 0$  for some  $x \in E$ . The conditional law of  $Y$  given  $X = x$ , denoted  $\mathcal{L}(Y | X = x)$ , is the probability measure on  $F$  with density (w.r.t.  $\beta$ )

$$g_x(y) = \frac{h(x, y)}{f(x)} = \frac{h(x, y)}{\int_F h(x, y) d\beta(y)}.$$

We may sometimes write  $g(y | x)$  instead of  $g_x(y)$  when there is no risk of confusion.

### Remark

For points where  $f(x) = 0$  we can define  $g_x$  arbitrarily (e.g. 0); these  $x$  typically form a set of  $\mathcal{L}(X)$ -measure zero, so they do not affect integrals.

# Random conditional density and joint density factorization

## Random conditional density

By extension, we define the **conditional density of  $Y$  given  $X$**  as the random density

$$g_{X,Y}(y) = g(y \mid X) = \begin{cases} \frac{h(X,y)}{f(X)}, & f(X) > 0, \\ 0, & f(X) = 0. \end{cases}$$

Since  $f(X) > 0$  almost surely, one usually just writes

$$g_{X,Y}(y) = \frac{h(X,y)}{f(X)}.$$

## Conditional expectation via conditional density

### Definition (conditional expectation)

Let  $\varphi : F \rightarrow \mathbb{R}$  be measurable with  $\mathbb{E}[\varphi(Y)] < \infty$ . The **conditional expectation** of  $\varphi(Y)$  given  $X$  is

$$\mathbb{E}[\varphi(Y) | X] = \int_F \varphi(y) g(y | X) d\beta(y).$$

This is a random variable measurable with respect to  $\sigma(X)$ .

### Law of total expectation

For any measurable  $\psi : E \times F \rightarrow \mathbb{R}$  such that  $\psi(X, Y)$  is integrable,

$$\mathbb{E}[\psi(X, Y)] = \mathbb{E}[\mathbb{E}[\psi(X, Y) | X]].$$

In particular, if  $\psi(X, Y) = \psi_1(X)\psi_2(Y)$  with integrable  $\psi_1(X)$  and  $\psi_2(Y)$ , then

$$\mathbb{E}[\psi_1(X)\psi_2(Y)] = \mathbb{E}[\psi_1(X) \mathbb{E}[\psi_2(Y) | X]].$$

## Conditional expectation as best $L^2$ predictor

### Projection property (orthogonality)

In the previous setting, let  $Y$  be square integrable:  $\mathbb{E}[Y^2] < \infty$ . Then

$$\inf_{\varphi: E \rightarrow \mathbb{R}, \mathbb{E}[\varphi(X)^2] < \infty} \mathbb{E}[(Y - \varphi(X))^2] = \mathbb{E}[(Y - \mathbb{E}[Y | X])^2].$$

Thus  $\mathbb{E}[Y | X]$  is the **best mean-square predictor** of  $Y$  among all (square integrable) functions of  $X$ .

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Thus  $\mathbb{E}[Y | X]$  is the **best mean-square predictor** of  $Y$  among all (square integrable) functions of  $X$ .

Proof:

For any measurable  $\varphi : E \rightarrow \mathbb{R}$  with  $\mathbb{E}[\varphi(X)^2] < \infty$ ,

$$\mathbb{E}[(Y - \varphi(X))^2] = \mathbb{E}[(Y - \mathbb{E}[Y | X])^2] + \mathbb{E}[(\mathbb{E}[Y | X] - \varphi(X))^2].$$

The cross-term is zero because

$$\mathbb{E}[(Y - \mathbb{E}[Y | X])(\mathbb{E}[Y | X] - \varphi(X))] = \mathbb{E}[\mathbb{E}[Y - \mathbb{E}[Y | X] | X](\mathbb{E}[Y | X] - \varphi(X))] = 0.$$

## Square integrability of $\mathbb{E}[Y | X]$

To justify the previous result we need  $\mathbb{E}[\mathbb{E}[Y | X]^2] < \infty$ .

By the conditional Jensen inequality,

$$\mathbb{E}[\mathbb{E}[Y | X]^2] \leq \mathbb{E}[\mathbb{E}[Y^2 | X]] = \mathbb{E}[Y^2] < \infty.$$

Hence  $\mathbb{E}[Y | X]$  is square integrable and the projection property makes sense in  $L^2$ .

## Frequentist approach

In the frequentist approach, we assume that there exists a *true but unknown* parameter value  $\theta_0 \in \Theta$  such that the data  $X$  follow the law  $P_{\theta_0}$ :

$$\exists \theta_0 \in \Theta \quad \text{s.t.} \quad X \sim P_{\theta_0}.$$

### Gaussian model

Let

$$X = (X_1, \dots, X_n), \quad \mathcal{P} = \{\mathcal{N}(\theta, 1)^{\otimes n} : \theta \in \mathbb{R}\}.$$

The frequentist assumption is that for some  $\theta_0 \in \mathbb{R}$ , the data are i.i.d.  $\mathcal{N}(\theta_0, 1)$ . One can then estimate  $\theta_0$  by the empirical mean  $\bar{X}_n$ ; by the law of large numbers,  $\bar{X}_n \xrightarrow{\mathbb{P}} \theta_0$ .

- **Estimation:** construct an estimator  $\hat{\theta}(X)$  close to  $\theta_0$ .
- **Confidence sets:** build random sets  $\mathcal{R}(X) \subset \Theta$  with  $\theta_0 \in \mathcal{R}(X)$  with high probability under  $P_{\theta_0}$ .
- **Tests:** answer "true/false" to a property of  $\theta_0$  via tests  $\varphi(X) \in \{0, 1\}$ .

## Bayesian approach: intuition

In the Bayesian approach, **all unknown quantities** are modeled as random variables.

### Prior and posterior

- Before observing data, our uncertainty about  $\theta$  is described by a **prior distribution**  $\Pi_0$  on  $\Theta$ .
- After observing  $X$ , we update this prior using Bayes' formula to obtain the **posterior distribution**  $\Pi(\cdot | X)$ .

The posterior combines:

- prior knowledge (or belief) about  $\theta$ ;
- the information contained in the data  $X$ .

### Coin tossing: frequentist vs Bayesian view

Let  $\theta \in [0, 1]$  be the probability of "heads".

- **Frequentist**:  $\theta$  is fixed; with many tosses, the empirical frequency  $\bar{X}_n$  converges to  $\theta$  (LLN, CLT).
- **Bayesian**: before any toss, we put a prior on  $\theta$  (e.g. uniform on  $[0, 1]$ ). Each new observation updates the prior to a posterior that reflects both prior belief and data.