UNIVERSITY OF TORONTO SCARBOROUGH

Department of Computer and Mathematical Sciences

STAC67H3F: Regression Analysis

Midterm Test

Saturday October 24, 2026 · Duration: 2.5 hours

Instructions

- Answer all questions in the spaces provided on this exam paper. This booklet is scanned: do not write on the QR codes at the top of the pages. If you run out of space, use the two pages for additional work before the tables, and say so in the space of the question.
- Show your work: a number with no supporting work receives no credit.
- Round your final answers to 2 decimal places unless the precision is stated.
- Some questions are meant to be *hard*. Do not spend an inordinate amount of time on any one question.
- There are 18 pages including this page, for a total of 100 points. The statistical tables (normal, t and F) are on the last three pages. Please check that you have all the pages.

Aids allowed

- One aid sheet, written on both sides, handwritten with a pen on paper (not typed, not printed, not written on a tablet and printed).
- Non-programmable and non-communicating calculator.

Good luck!

Exercise 1 (14 points in total)

Short answer questions.

(a) (8 points) For each of the following statements, specify either TRUE or FALSE, and explain why.

(i) In a multiple regression model with an intercept, fitted by least squares, the residuals $e_1, \ldots, e_n$ are independent random variables, each with variance σ^2. [4]

(ii) In simple linear regression, the F test, built with the ANOVA table, and the two-sided t test of $H_0 : \beta_1 = 0$, carried out at the same level α, always lead to the same conclusion. [4]

(b) (6 points) Consider the simple linear regression model $Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$, $i = 1, \ldots, n$, where the ε_i are i.i.d. $N(0, 3^2)$ and the x_i are known constants. Suppose that we have $n = 3$ observations in the response vector $\boldsymbol{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$. Let $W_1 = Y_1 - Y_2$ and $W_2 = Y_1 + Y_2 - 2Y_3$.

(i) Find the constant matrix $\boldsymbol{A}$ such that $\boldsymbol{W} = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \boldsymbol{AY}$, and use it to calculate the variance-covariance matrix of the random vector $\boldsymbol{W}$. [3]

(ii) Are W_1 and W_2 independent? Justify your answer. [3]

Exercise 2 (22 points in total)

The employees of $n = 30$ departments of a large company answered a survey. For each department, the response variable is the overall rating of the department (Y, the percentage of favourable answers), and the predictor is the rating of how the department handles employee complaints (x, also a percentage of favourable answers). The values of x range from 37 to 90. A simple linear regression model is fitted, and the estimated regression coefficient vector and its estimated variance-covariance matrix are given below:

$$\hat{\boldsymbol{\beta}} = \begin{bmatrix} 14.376 \\ 0.7546 \end{bmatrix}, \qquad \boldsymbol{s^2}\{\hat{\boldsymbol{\beta}}\} = \begin{bmatrix} 43.824 & -0.63354 \\ -0.63354 & 0.0095127 \end{bmatrix}.$$

(a) Obtain the predicted overall rating of a department whose complaints rating is 80. [2]

(b) Compute a 95% confidence interval for the mean overall rating of all departments whose complaints rating is 80. [8]

(c) A manager claims that each additional point of complaints rating comes with a full additional point of overall rating, on average. Test $H_0 : \beta_1 = 1$ against $H_a : \beta_1 \neq 1$ with $\alpha = 0.05$. [6]

(d) The residual standard error of the model is 6.993. Construct a 95% prediction interval for the overall rating of one department whose complaints rating is 80. [6]

Exercise 3 (28 points in total)

The dataset `mtcars` gives the following variables for $n = 32$ car models:

- **mpg**: the fuel efficiency, in miles per gallon;
- **wt**: the weight of the car, in thousands of pounds;
- **hp**: the engine power, in horsepower.

First, we fit the simple regression model to predict **mpg** from **wt**. Part of the R output is as follows:

```
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  37.2851     1.8776  19.858  < 2e-16 ***
wt           -5.3445        (b)  -9.559 1.29e-10 ***
---
Multiple R-squared:  (a),    Adjusted R-squared:  xxxx
```

(a) Fill out (a) and (b), giving (b) to 4 decimal places, and interpret the value (a). [6]

Now, we fit a multiple regression model to predict **mpg** from **wt** and **hp**.

```
> fit <- lm(mpg ~ wt + hp, data = mtcars)
> summary(fit)
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 37.22727    1.59879  23.285  < 2e-16 ***
wt          -3.87783    0.63273  -6.129 1.12e-06 ***
hp          -0.03177    0.00903  -3.519  0.00145 **
---
Residual standard error: 2.593 on xx degrees of freedom
Multiple R-squared:  xxxx,    Adjusted R-squared:  xxxx
F-statistic: xxxx on xx and xx DF,  p-value: 9.109e-12

> new.data <- data.frame(wt = 3, hp = 150)
> round(predict(fit, new.data, interval = "confidence"), 2)
    fit   lwr   upr
1 20.83 19.84 21.82
```

(b) From this output, construct a 95% prediction interval for the fuel efficiency of one car weighing 3000 pounds with an engine of 150 horsepower. (The interval printed by R above has level 95%.) [7]

(c) (9 points) The total sum of squares of **mpg** is $SST = 1126.05$.

(i) Based on this, compute the Multiple R-squared, the Adjusted R-squared and the value of the F statistic that are missing from the R output of the multiple regression model. [5]

(ii) Perform the global F test of the multiple regression model, with $\alpha = 0.05$. [4]

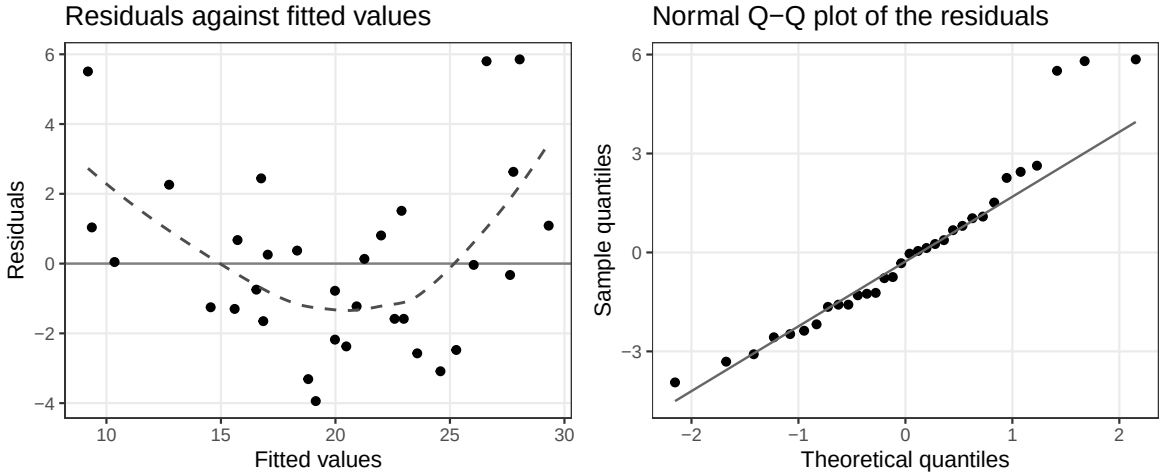
- Write down the hypotheses:

  H_0: _____ H_a: _____

- Provide the critical value when $\alpha = 0.05$: _____

- Make a conclusion.

(d) The two graphs and the two tests below are for the residuals of the multiple regression model (the dashed curve is a lowess curve). Check the assumptions of the regression model based on the graphs and the tests. If an assumption is violated, propose a remedial method to fix the problem. [6]




```
Brown-Forsythe test (two groups, split at the median of the fitted values)
t = -0.568, df = 30, p-value = 0.5742

Shapiro-Wilk normality test
W = 0.92792, p-value = 0.03427
```

Exercise 4 (16 points in total)

A city's water utility recorded, for $n = 25$ summer days, the total water consumption of the city (Y, in megalitres), the maximum temperature of the day (X_1, in degrees Celsius) and the rainfall of the day (X_2, in millimetres). Over these days, X_1 ranges from 19.2 to 33.5 and X_2 from 1.0 to 20.2. The regression model $Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \varepsilon_i$, with independent $N(0, \sigma^2)$ errors, is fitted. The following results are obtained:

$$(\boldsymbol{X}'\boldsymbol{X})^{-1} = \begin{bmatrix} 1.401 & -0.05277 & 0.002155 \\ -0.05277 & 0.002145 & -0.0005726 \\ 0.002155 & -0.0005726 & 0.002419 \end{bmatrix}, \qquad \hat{\boldsymbol{\beta}} = \begin{bmatrix} 56.514 \\ 2.704 \\ -1.345 \end{bmatrix}, \qquad SSE = 904.20.$$

(a) Compute a 95% confidence interval for β_2, and interpret it in context. What does this interval tell you about the test of $H_0 : \beta_2 = 0$ against $H_a : \beta_2 \neq 0$ at the 5% level? [7]

(b) Construct a 95% prediction interval for the water consumption of the city on one day with a maximum temperature of 38 degrees and 5 millimetres of rain. Can this interval be interpreted as a 95% prediction interval in the usual way? Explain. [9]

Exercise 5 (20 points in total)

Consider the regression model through the origin

$$Y_i = \beta x_i + \varepsilon_i, \qquad i = 1, \ldots, n, \qquad \text{where } \varepsilon_i \overset{\text{iid}}{\sim} N(0, \sigma^2),$$

and the x_i are known constants, not all equal to zero. Write $\boldsymbol{x} = (x_1, \ldots, x_n)'$.

(a) Write this model in matrix form: what are the design matrix $\boldsymbol{X}$ and the parameter vector? How many parameters does the model have? [2]

(b) Compute $\boldsymbol{X}'\boldsymbol{X}$ and $\boldsymbol{X}'\boldsymbol{Y}$, and use $\hat{\boldsymbol{\beta}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y}$ to find $\hat{\beta}$. Then show that the hat matrix is $\boldsymbol{H} = \boldsymbol{x}\boldsymbol{x}'/\sum_{k=1}^{n} x_k^2$, give its diagonal entries h_{ii}, and show that $\sum_{i=1}^{n} h_{ii} = 1$. [7]

(c) Find $E(\hat{\beta})$ and $\text{Var}(\hat{\beta})$. What is the sampling distribution of $\hat{\beta}$? [4]

(d) Using the result $E(\boldsymbol{Y}'\boldsymbol{A}\boldsymbol{Y}) = \text{tr}(\boldsymbol{A}\boldsymbol{\Sigma}) + \boldsymbol{\mu}'\boldsymbol{A}\boldsymbol{\mu}$, valid for a symmetric constant matrix $\boldsymbol{A}$ when $E(\boldsymbol{Y}) = \boldsymbol{\mu}$ and $\boldsymbol{\sigma^2}\{\boldsymbol{Y}\} = \boldsymbol{\Sigma}$, show that $E(SSE) = (n-1)\,\sigma^2$, where $SSE = \boldsymbol{Y}'(\boldsymbol{I} - \boldsymbol{H})\boldsymbol{Y}$. Deduce an unbiased estimator of σ^2. [7]

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Standard normal distribution

Entries are $P(Z \leq z)$ for $Z \sim N(0,1)$. For negative z, use $P(Z \leq -z) = 1 - P(Z \leq z)$.

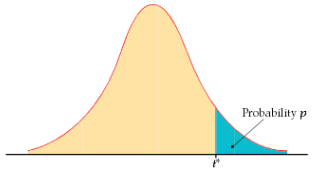
z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Quantiles z_p, where $P(Z \leq z_p) = p$:

p	0.900	0.950	0.975	0.990	0.995
z_p	1.282	1.645	1.960	2.326	2.576

t-distribution table

Areas in the upper tail are given along the top of the table. Critical t* values are given in the table.



df	0.1	0.05	0.025	0.02	0.01	0.005
1	3.078	6.314	12.706	15.895	31.821	63.657
2	1.886	2.920	4.303	4.849	6.965	9.925
3	1.638	2.353	3.182	3.482	4.541	5.841
4	1.533	2.132	2.776	2.999	3.747	4.604
5	1.476	2.015	2.571	2.757	3.365	4.032
6	1.440	1.943	2.447	2.612	3.143	3.707
7	1.415	1.895	2.365	2.517	2.998	3.499
8	1.397	1.860	2.306	2.449	2.896	3.355
9	1.383	1.833	2.262	2.398	2.821	3.250
10	1.372	1.812	2.228	2.359	2.764	3.169
11	1.363	1.796	2.201	2.328	2.718	3.106
12	1.356	1.782	2.179	2.303	2.681	3.055
13	1.350	1.771	2.160	2.282	2.650	3.012
14	1.345	1.761	2.145	2.264	2.624	2.977
15	1.341	1.753	2.131	2.249	2.602	2.947
16	1.337	1.746	2.120	2.235	2.583	2.921
17	1.333	1.740	2.110	2.224	2.567	2.898
18	1.330	1.734	2.101	2.214	2.552	2.878
19	1.328	1.729	2.093	2.205	2.539	2.861
20	1.325	1.725	2.086	2.197	2.528	2.845
21	1.323	1.721	2.080	2.189	2.518	2.831
22	1.321	1.717	2.074	2.183	2.508	2.819
23	1.319	1.714	2.069	2.177	2.500	2.807
24	1.318	1.711	2.064	2.172	2.492	2.797
25	1.316	1.708	2.060	2.167	2.485	2.787
26	1.315	1.706	2.056	2.162	2.479	2.779
27	1.314	1.703	2.052	2.158	2.473	2.771
28	1.313	1.701	2.048	2.154	2.467	2.763
29	1.311	1.699	2.045	2.150	2.462	2.756
30	1.310	1.697	2.042	2.147	2.457	2.750
31	1.309	1.696	2.040	2.144	2.453	2.744
32	1.309	1.694	2.037	2.141	2.449	2.738
33	1.308	1.692	2.035	2.138	2.445	2.733
34	1.307	1.691	2.032	2.136	2.441	2.728
35	1.306	1.690	2.030	2.133	2.438	2.724
36	1.306	1.688	2.028	2.131	2.434	2.719
37	1.305	1.687	2.026	2.129	2.431	2.715
38	1.304	1.686	2.024	2.127	2.429	2.712
39	1.304	1.685	2.023	2.125	2.426	2.708
40	1.303	1.684	2.021	2.123	2.423	2.704
41	1.303	1.683	2.020	2.121	2.421	2.701
42	1.302	1.682	2.018	2.120	2.418	2.698
43	1.302	1.681	2.017	2.118	2.416	2.695
44	1.301	1.680	2.015	2.116	2.414	2.692
45	1.301	1.679	2.014	2.115	2.412	2.690
46	1.300	1.679	2.013	2.114	2.410	2.687
47	1.300	1.678	2.012	2.112	2.408	2.685
48	1.299	1.677	2.011	2.111	2.407	2.682
49	1.299	1.677	2.010	2.110	2.405	2.680
50	1.299	1.676	2.009	2.109	2.403	2.678

df	0.1	0.05	0.025	0.02	0.01	0.005
51	1.298	1.675	2.008	2.108	2.402	2.676
52	1.298	1.675	2.007	2.107	2.400	2.674
53	1.298	1.674	2.006	2.106	2.399	2.672
54	1.297	1.674	2.005	2.105	2.397	2.670
55	1.297	1.673	2.004	2.104	2.396	2.668
56	1.297	1.673	2.003	2.103	2.395	2.667
57	1.297	1.672	2.002	2.102	2.394	2.665
58	1.296	1.672	2.002	2.101	2.392	2.663
59	1.296	1.671	2.001	2.100	2.391	2.662
60	1.296	1.671	2.000	2.099	2.390	2.660
61	1.296	1.670	2.000	2.099	2.389	2.659
62	1.295	1.670	1.999	2.098	2.388	2.657
63	1.295	1.669	1.998	2.097	2.387	2.656
64	1.295	1.669	1.998	2.096	2.386	2.655
65	1.295	1.669	1.997	2.096	2.385	2.654
66	1.295	1.668	1.997	2.095	2.384	2.652
67	1.294	1.668	1.996	2.095	2.383	2.651
68	1.294	1.668	1.995	2.094	2.382	2.650
69	1.294	1.667	1.995	2.093	2.382	2.649
70	1.294	1.667	1.994	2.093	2.381	2.648
71	1.294	1.667	1.994	2.092	2.380	2.647
72	1.293	1.666	1.993	2.092	2.379	2.646
73	1.293	1.666	1.993	2.091	2.379	2.645
74	1.293	1.666	1.993	2.091	2.378	2.644
75	1.293	1.665	1.992	2.090	2.377	2.643
76	1.293	1.665	1.992	2.090	2.376	2.642
77	1.293	1.665	1.991	2.089	2.376	2.641
78	1.292	1.665	1.991	2.089	2.375	2.640
79	1.292	1.664	1.990	2.088	2.374	2.640
80	1.292	1.664	1.990	2.088	2.374	2.639
81	1.292	1.664	1.990	2.087	2.373	2.638
82	1.292	1.664	1.989	2.087	2.373	2.637
83	1.292	1.663	1.989	2.087	2.372	2.636
84	1.292	1.663	1.989	2.086	2.372	2.636
85	1.292	1.663	1.988	2.086	2.371	2.635
86	1.291	1.663	1.988	2.085	2.370	2.634
87	1.291	1.663	1.988	2.085	2.370	2.634
88	1.291	1.662	1.987	2.085	2.369	2.633
89	1.291	1.662	1.987	2.084	2.369	2.632
90	1.291	1.662	1.987	2.084	2.368	2.632
91	1.291	1.662	1.986	2.084	2.368	2.631
92	1.291	1.662	1.986	2.083	2.368	2.630
93	1.291	1.661	1.986	2.083	2.367	2.630
94	1.291	1.661	1.986	2.083	2.367	2.629
95	1.291	1.661	1.985	2.082	2.366	2.629
96	1.290	1.661	1.985	2.082	2.366	2.628
97	1.290	1.661	1.985	2.082	2.365	2.627
98	1.290	1.661	1.984	2.081	2.365	2.627
99	1.290	1.660	1.984	2.081	2.365	2.626
100	1.290	1.660	1.984	2.081	2.364	2.626

Critical Values of the F-Distribution: $\alpha = 0.05$

Denom.	Numerator Degrees of Freedom									
d.f.	1	2	3	4	5	6	7	8	9	10
1	161.448	199.500	215.707	224.583	230.162	233.986	236.768	238.883	240.543	241.882
2	18.513	19.000	19.164	19.247	19.296	19.330	19.353	19.371	19.385	19.396
3	10.128	9.552	9.277	9.117	9.013	8.941	8.887	8.845	8.812	8.786
4	7.709	6.944	6.591	6.388	6.256	6.163	6.094	6.041	5.999	5.964
5	6.608	5.786	5.409	5.192	5.050	4.950	4.876	4.818	4.772	4.735
6	5.987	5.143	4.757	4.534	4.387	4.284	4.207	4.147	4.099	4.060
7	5.591	4.737	4.347	4.120	3.972	3.866	3.787	3.726	3.677	3.637
8	5.318	4.459	4.066	3.838	3.687	3.581	3.500	3.438	3.388	3.347
9	5.117	4.256	3.863	3.633	3.482	3.374	3.293	3.230	3.179	3.137
10	4.965	4.103	3.708	3.478	3.326	3.217	3.135	3.072	3.020	2.978
11	4.844	3.982	3.587	3.357	3.204	3.095	3.012	2.948	2.896	2.854
12	4.747	3.885	3.490	3.259	3.106	2.996	2.913	2.849	2.796	2.753
13	4.667	3.806	3.411	3.179	3.025	2.915	2.832	2.767	2.714	2.671
14	4.600	3.739	3.344	3.112	2.958	2.848	2.764	2.699	2.646	2.602
15	4.543	3.682	3.287	3.056	2.901	2.790	2.707	2.641	2.588	2.544
16	4.494	3.634	3.239	3.007	2.852	2.741	2.657	2.591	2.538	2.494
17	4.451	3.592	3.197	2.965	2.810	2.699	2.614	2.548	2.494	2.450
18	4.414	3.555	3.160	2.928	2.773	2.661	2.577	2.510	2.456	2.412
19	4.381	3.522	3.127	2.895	2.740	2.628	2.544	2.477	2.423	2.378
20	4.351	3.493	3.098	2.866	2.711	2.599	2.514	2.447	2.393	2.348
21	4.325	3.467	3.072	2.840	2.685	2.573	2.488	2.420	2.366	2.321
22	4.301	3.443	3.049	2.817	2.661	2.549	2.464	2.397	2.342	2.297
23	4.279	3.422	3.028	2.796	2.640	2.528	2.442	2.375	2.320	2.275
24	4.260	3.403	3.009	2.776	2.621	2.508	2.423	2.355	2.300	2.255
25	4.242	3.385	2.991	2.759	2.603	2.490	2.405	2.337	2.282	2.236
26	4.225	3.369	2.975	2.743	2.587	2.474	2.388	2.321	2.265	2.220
27	4.210	3.354	2.960	2.728	2.572	2.459	2.373	2.305	2.250	2.204
28	4.196	3.340	2.947	2.714	2.558	2.445	2.359	2.291	2.236	2.190
29	4.183	3.328	2.934	2.701	2.545	2.432	2.346	2.278	2.223	2.177
30	4.171	3.316	2.922	2.690	2.534	2.421	2.334	2.266	2.211	2.165
31	4.160	3.305	2.911	2.679	2.523	2.409	2.323	2.255	2.199	2.153
32	4.149	3.295	2.901	2.668	2.512	2.399	2.313	2.244	2.189	2.142
33	4.139	3.285	2.892	2.659	2.503	2.389	2.303	2.235	2.179	2.133
34	4.130	3.276	2.883	2.650	2.494	2.380	2.294	2.225	2.170	2.123
35	4.121	3.267	2.874	2.641	2.485	2.372	2.285	2.217	2.161	2.114
36	4.113	3.259	2.866	2.634	2.477	2.364	2.277	2.209	2.153	2.106
37	4.105	3.252	2.859	2.626	2.470	2.356	2.270	2.201	2.145	2.098
38	4.098	3.245	2.852	2.619	2.463	2.349	2.262	2.194	2.138	2.091
39	4.091	3.238	2.845	2.612	2.456	2.342	2.255	2.187	2.131	2.084
40	4.085	3.232	2.839	2.606	2.449	2.336	2.249	2.180	2.124	2.077
41	4.079	3.226	2.833	2.600	2.443	2.330	2.243	2.174	2.118	2.071
42	4.073	3.220	2.827	2.594	2.438	2.324	2.237	2.168	2.112	2.065
43	4.067	3.214	2.822	2.589	2.432	2.318	2.232	2.163	2.106	2.059
44	4.062	3.209	2.816	2.584	2.427	2.313	2.226	2.157	2.101	2.054
45	4.057	3.204	2.812	2.579	2.422	2.308	2.221	2.152	2.096	2.049
46	4.052	3.200	2.807	2.574	2.417	2.304	2.216	2.147	2.091	2.044
47	4.047	3.195	2.802	2.570	2.413	2.299	2.212	2.143	2.086	2.039
48	4.043	3.191	2.798	2.565	2.409	2.295	2.207	2.138	2.082	2.035
49	4.038	3.187	2.794	2.561	2.404	2.290	2.203	2.134	2.077	2.030
50	4.034	3.183	2.790	2.557	2.400	2.286	2.199	2.130	2.073	2.026
60	4.001	3.150	2.758	2.525	2.368	2.254	2.167	2.097	2.040	1.993
70	3.978	3.128	2.736	2.503	2.346	2.231	2.143	2.074	2.017	1.969
80	3.960	3.111	2.719	2.486	2.329	2.214	2.126	2.056	1.999	1.951
90	3.947	3.098	2.706	2.473	2.316	2.201	2.113	2.043	1.986	1.938
100	3.936	3.087	2.696	2.463	2.305	2.191	2.103	2.032	1.975	1.927
120	3.920	3.072	2.680	2.447	2.290	2.175	2.087	2.016	1.959	1.910
140	3.909	3.061	2.669	2.436	2.279	2.164	2.076	2.005	1.947	1.899
180	3.894	3.046	2.655	2.422	2.264	2.149	2.061	1.990	1.932	1.884
200	3.888	3.041	2.650	2.417	2.259	2.144	2.056	1.985	1.927	1.878
∞	3.841	2.996	2.605	2.372	2.214	2.099	2.010	1.938	1.880	1.831