

STAC67: Regression Analysis

Lecture 5

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Where we stopped last week

Everything so far used **only the first two moments** of the noise:

- $E(\epsilon_i) = 0$,
- $Var(\epsilon_i) = \sigma^2$,
- $Cov(\epsilon_i, \epsilon_j) = 0$ for $i \neq j$.

We obtained the properties of least-square estimators:

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

- $\hat{\beta}_1 = \frac{S_{xY}}{S_{xx}} = \sum_{i=1}^n k_i Y_i$ with $k_i = \frac{x_i - \bar{x}}{S_{xx}}$ is **linear in the Y_i** ,
- $E(\hat{\beta}_0) = \beta_0$, $E(\hat{\beta}_1) = \beta_1$: **unbiased**,
- $Var(\hat{\beta}_1) = \frac{\sigma^2}{S_{xx}}$ and $Var(\hat{\beta}_0) = \sigma^2 \left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}} \right)$
- **Gauss-Markov**: among all linear unbiased estimators, both have the smallest variance (**BLUE**)
- σ^2 is estimated without bias by $\hat{\sigma} = MSE = \frac{SSE}{n-2} = \frac{\sum_i e_i^2}{n-2}$, and

$$SE(\hat{\beta}_1) = \frac{\hat{\sigma}}{\sqrt{S_{xx}}}, \quad SE(\hat{\beta}_0) = \hat{\sigma}^2 \left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}} \right)$$

Inference about Regression Parameters

What we cannot do yet is make any probabilistic statement about the estimator.

Mean and variance do not say what is the probability $\hat{\beta}_j$ is to land within a given distance of β_j . Reminder: we can obtain **bounds** with Markov's inequality.

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So recall the simple Linear Model with **Normal Errors**

$$Y_i = \beta_0 + \beta_1 X_i + \epsilon_i,$$

where

- β_0 and β_1 are parameters
- ϵ_i are i.i.d with **normal distribution** with mean 0 and variance, σ^2 .

$$\epsilon_i \sim N(0, \sigma^2)$$

- Under this model,

$$Y_i | X_i \sim N(\beta_0 + \beta_1 X_i, \sigma^2)$$

Inference about Regression Parameters

- Inference about β_1 *iid*

$$\hat{\beta}_1 = \sum_i \beta_1 y_i \sim N(\beta_1, \frac{\sigma^2}{S_{xx}}) \Leftrightarrow \frac{\hat{\beta}_1 - \beta_1}{\sigma / \sqrt{S_{xx}}} \sim N(0, 1)$$

- σ is unknown, and so we can replace it with $\hat{\sigma}$. In this case, we have

$$T = \frac{\hat{\beta}_1 - \beta_1}{\left(\frac{\hat{\sigma}}{\sqrt{S_{xx}}} \right)} \sim t_{n-2}$$

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where $Z \sim N(0, 1)$, $V \sim \chi_k^2$, and Z and V are **independent**.

$$16 \quad w_1, \dots, w_k \stackrel{\text{iid}}{\sim} N(0, 1), \quad \sum_{i=1}^k w_i^2 \sim \chi_k^2$$

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So to prove the student distribution we need three things: 1) a normal numerator, 2) a chi-square in the denominator, and 3) independence between them.

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3. **Admitted** The denominator is a chi-square over its degrees of freedom:

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4. **Admitted** The two are independent. ($Cov(\hat{\beta}_1, e_i) = 0$, and under normality zero covariance gives independence.)

Inference about Regression Parameters

Theorem 3 (Sampling distributions)

$$\frac{\hat{\beta}_0 - \beta_0}{SE(\hat{\beta}_0)} \sim t_{n-2}, \quad \frac{\hat{\beta}_1 - \beta_1}{SE(\hat{\beta}_1)} \sim t_{n-2}$$

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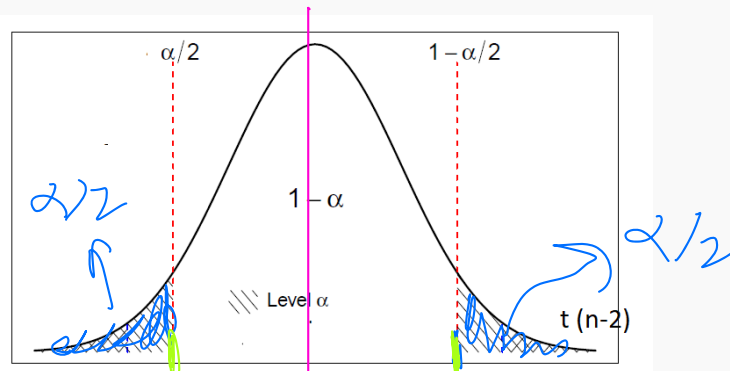
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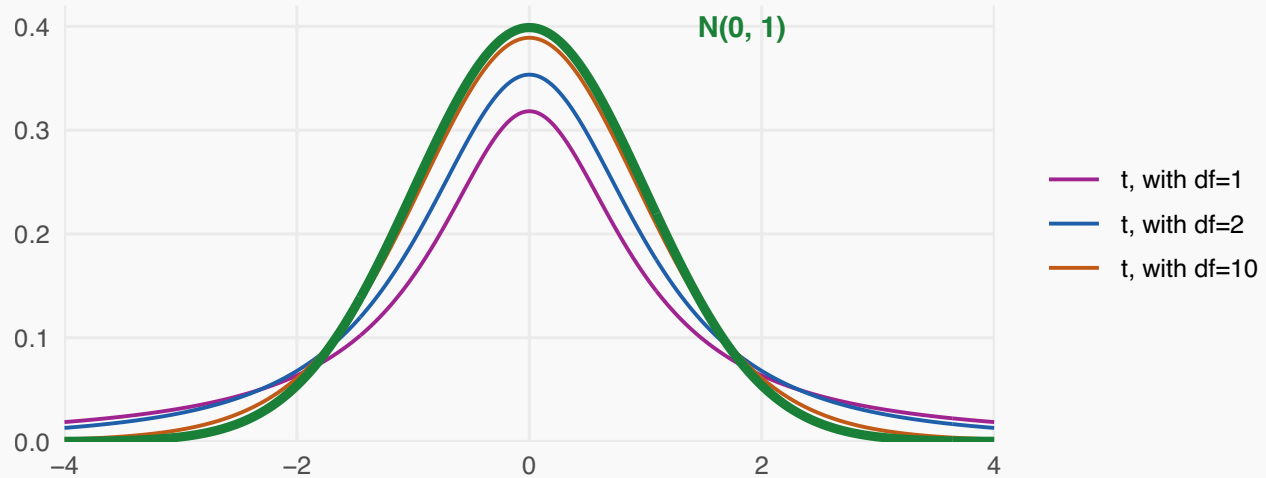
quantile at level $1 - \alpha/2$
of t_{n-2}

$$P\left(-t(1 - \alpha/2; n - 2) \leq \frac{\hat{\beta}_j - \beta_j}{SE(\hat{\beta}_j)} \leq t(1 - \alpha/2; n - 2)\right) = 1 - \alpha$$

$t(\alpha/2; n-2) \leftarrow \text{Ⓢ}$
 $= -t(1 - \alpha/2; n-2)$

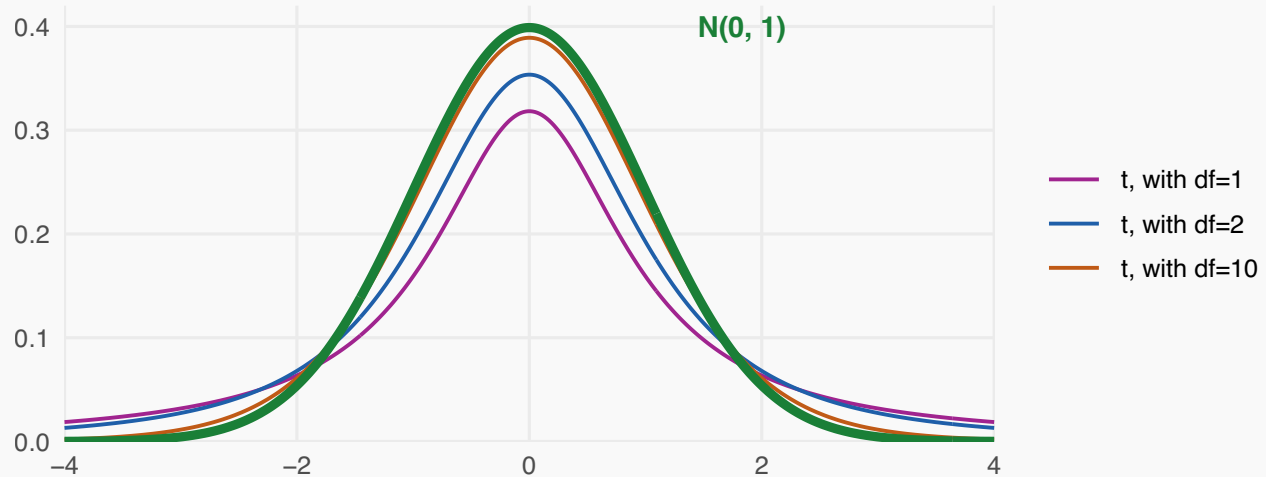


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Heavier tails than the normal: larger quantiles are the price of estimating σ rather than knowing it.
The difference vanishes as the degrees of freedom grow.

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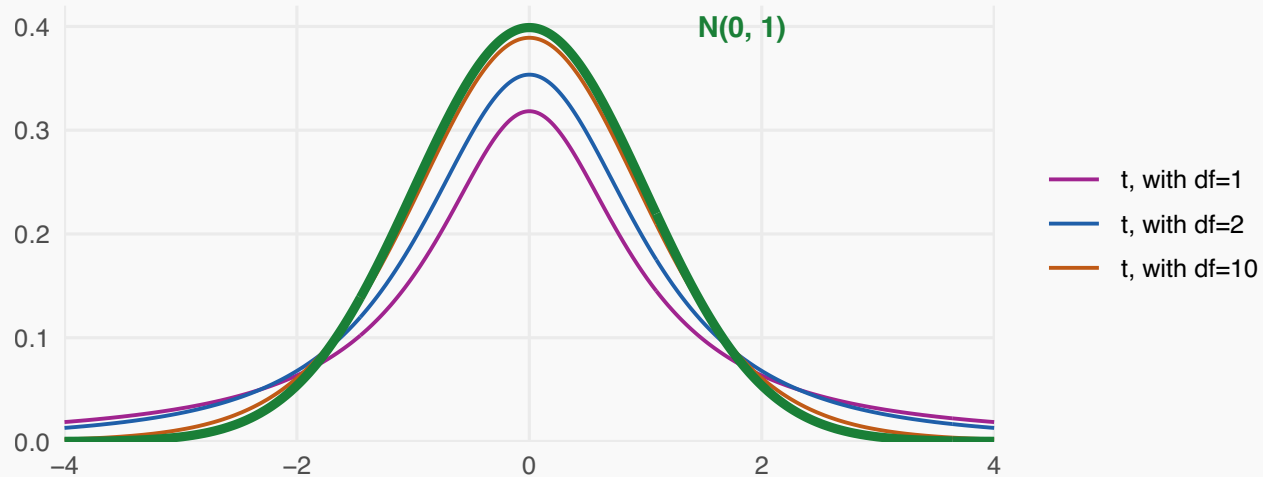


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⚠ Be careful with small sample sizes as in this course! With $n = 24$, we use $t(0.975; 22) = 2.074$, and 1.96 would make the interval **6% too narrow**. This is why R always reports t intervals, whatever n is.

$$\underbrace{-t(1 - \alpha/2; n - 2)}_{\text{lower bound}} \leq \frac{\hat{\beta}_j - \beta_j}{SE(\hat{\beta}_j)} \leq \underbrace{t(1 - \alpha/2; n - 2)}_{\text{upper bound}} \quad \Rightarrow \quad \beta_j = \hat{\beta}_j \pm t(1 - \alpha/2; n - 2) SE$$

Confidence Intervals

$$\hat{\beta}_j \pm t(1 - \alpha/2; n - 2)SE(\hat{\beta}_j), \quad j = 0, 1$$

Why should we care about these confidence intervals?

Because they **completely** capture how much information in the data we have about the parameter.

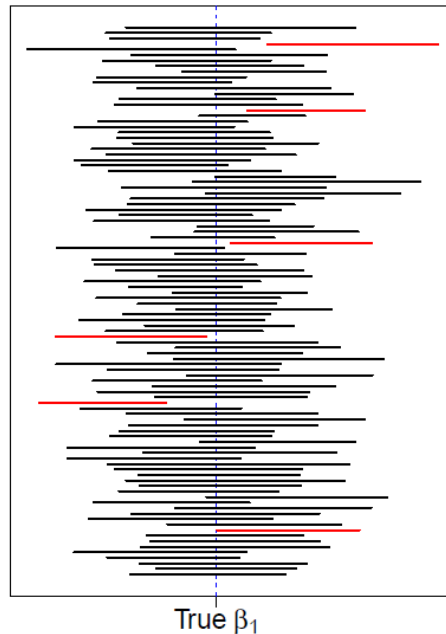
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The $1 - \alpha$ confidence level, over repeated samples, the proportion of the intervals build this way that contain β_j . The one you build from a particular dataset either does or does not.



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3. Decision Rule:

- P-value approach: we can compute

$$\text{p-value} = 2 \times P(t_{n-2} \geq |t^*|)$$

Fix α and
compute t^*

If **p-value** $< \alpha$, **reject** H_0 ; otherwise **do not reject** H_0 ,

where α is the level of significance, or probability of **type I error**, $\alpha = \Pr(\text{Reject } H_0 \mid H_0 \text{ is true})$.

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- Critical value approach:

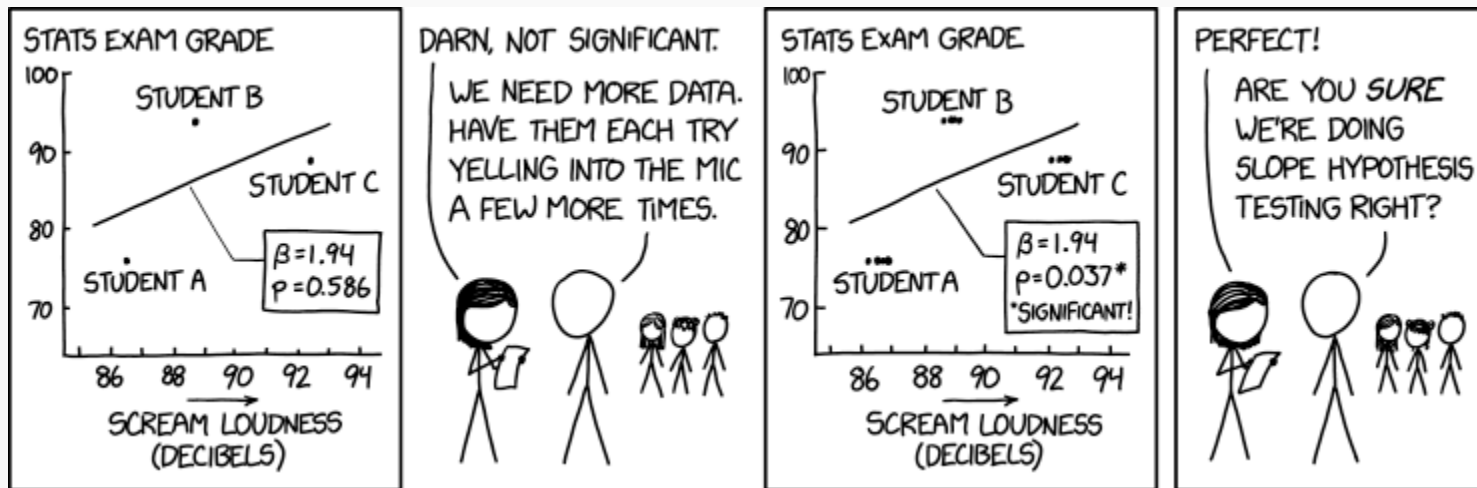
If $|t^*| > t(1 - \alpha/2; n - 2)$, **reject** H_0 .

Compute t^* and
 $t(1 - \alpha/2; n - 2)$

If $|t^*| \leq t(1 - \alpha/2; n - 2)$, **do not reject** H_0 .

↪ equivalent to " β_1^0 not in the confidence interval"

On the importance of assumptions



The estimate never moves: $\hat{\beta}_1 = 1.94$ in both panels. Only the p -value does, because the extra points are repeated measurements on the same three students and add more confidence in the estimates.

xkcd.com/2533 (CC BY-NC 2.5)

Exercise (Crime Rate)

Let's go back to the example from last week.

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	20517.5999	3277.64269	6.259865	0.00e+00
X	-170.5752	41.57433	-4.102897	9.57e-05

- We want to test whether or not there is a linear association between crime rate and percentage of high school graduates, using a t test with $\alpha = 0.01$. State the hypotheses, decision rule, and conclusion.

Exercise (Crime Rate)

- Estimate β_1 with a 99 % confidence interval. Interpret the interval estimate.