STAC67: Regression Analysis

Lecture 5

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Where we stopped last week

Everything so far used **only the first two moments** of the noise:

- $E(\epsilon_i) = 0$,
- $Var(\epsilon_i) = \sigma^2$,
- $Cov(\epsilon_i, \epsilon_j) = 0$ for $i \neq j$.

We obtained the properties of least-square estimators:

- $\hat{\beta}_1 = \frac{S_{xY}}{S_{xx}} = \sum_{i=1}^{n} k_i Y_i$ with $k_i = \frac{x_i - \bar{x}}{S_{xx}}$ is linear in the Y_i,
- $E(\hat{\beta}_0) = \beta_0$, $E(\hat{\beta}_1) = \beta_1$: unbiased,
- $Var(\hat{\beta}_1) = \frac{\sigma^2}{S_{xx}}$ and $Var(\hat{\beta}_0) = \sigma^2 \left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}} \right)$
- **Gauss-Markov**: among all linear unbiased estimators, both have the smallest variance (BLUE)
- σ^2 is estimated without bias by $\hat{\sigma} = MSE = \frac{SSE}{n-2} = \frac{\sum_i e_i^2}{n-2}$, and

$$SE(\hat{\beta}_1) = \frac{\hat{\sigma}}{\sqrt{S_{xx}}}, \quad SE(\hat{\beta}_0) = \hat{\sigma}^2 \left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}} \right)$$

Inference about Regression Parameters

What we cannot do yet is make any probabilistic statement about the estimator.

Mean and variance do not say what is the probability $\hat{\beta}_j$ is to land within a given distance of β_j. Reminder: we can obtain **bounds** with Markov's inequality.

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So recall the simple Linear Model with **Normal Errors**

$$Y_i = \beta_0 + \beta_1 X_i + \epsilon_i,$$

where

- β_0 and β_1 are parameters
- ϵ_i are i.i.d with normal distribution with mean 0 and variance, σ^2.

$$\epsilon_i \sim N(0, \sigma^2)$$

- Under this model,

$$Y_i | X_i \sim N(\beta_0 + \beta_1 X_i, \sigma^2)$$

Inference about Regression Parameters

- Inference about β_1

- σ is unknown, and so we can replace it with $\hat{\sigma}$. In this case, we have

$$T = \frac{\hat{\beta}_1 - \beta_1}{\left(\frac{\hat{\sigma}}{\sqrt{S_{xx}}}\right)} \sim t_{n-2}$$

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So to prove the student distribution we need three things: 1) a normal numerator, 2) a chi-square in the denominator, and 3) independence between them.

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3. **Admitted** The denominator is a chi-square over its degrees of freedom:

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4. **Admitted** The two are independent. ($Cov(\hat{\beta}_1, e_i) = 0$, and under normality zero covariance gives independence.)

Inference about Regression Parameters

Theorem 3 (Sampling distributions)

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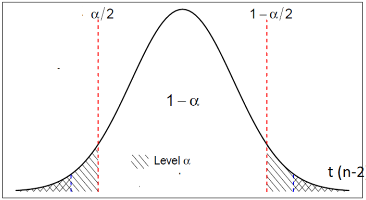
$$P\left(-t(1-\alpha/2; n-2) \leq \frac{\hat{\beta}_j - \beta_j}{SE(\hat{\beta}_j)} \leq t(1-\alpha/2; n-2)\right) = 1 - \alpha$$



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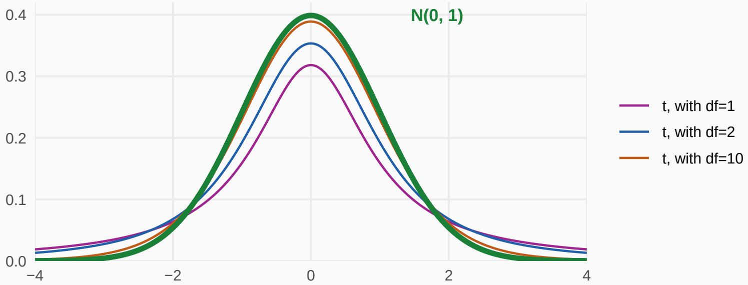

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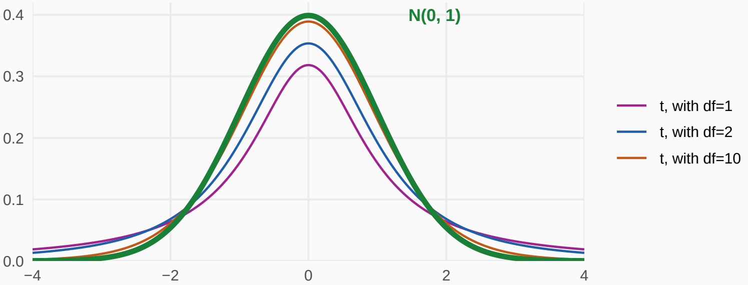

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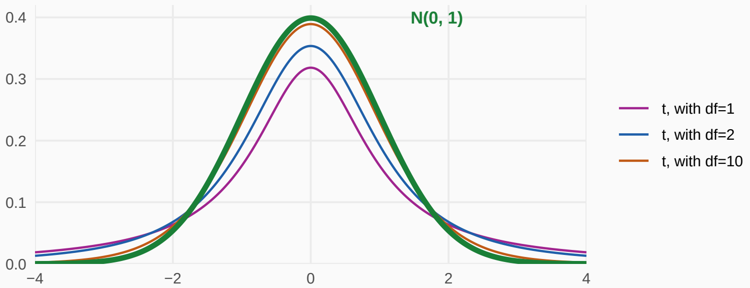

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⚠ Be careful with small sample sizes as in this course! With $n = 24$, we use $t(0.975;\ 22) = 2.074$, and 1.96 would make the interval **6% too narrow**. This is why R always reports t intervals, whatever n is.

Confidence Intervals

$$\hat{\beta}_j \pm t(1-\alpha/2; n-2)SE(\hat{\beta}_j), \quad j = 0, 1$$

Why should we care about these confidence intervals?

Because they completely capture how much information in the data we have about the parameter.

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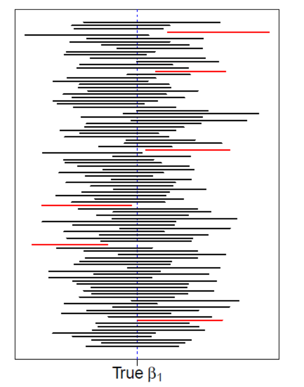
The $1-\alpha$ confidence level, over repeated samples, the proportion of the intervals build this wat that contain β_j. The one you build from a particular dataset either does or does not.



Hypothesis test concerning β_1

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3. **Decision Rule**:

   - P-value approach: we can compute

     $$\text{p-value} = 2 \times P(t_{n-2} \geq |t^*|)$$

     If p-value $< \alpha$, **reject** H_0; otherwise **do not reject** H_0,

     where α is the level of significance, or probability of type I error, $\alpha =$ Pr(Reject H_0 | H_0 is true).

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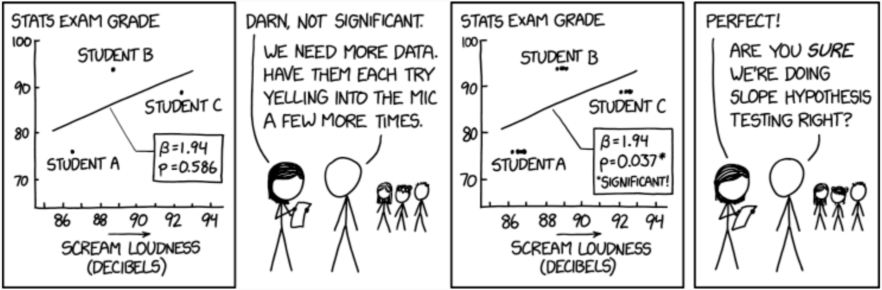
   - Critical value approach:

     If $|t^*| > t(1 - \alpha/2; n - 2)$, **reject** H_0.

     If $|t^*| \leq t(1 - \alpha/2; n - 2)$, **do not reject** H_0.

On the importance of assumptions



The estimate never moves: $\hat{\beta}_1 = 1.94$ in both panels. Only the p-value does, because the extra points are repeated measurements on the same three students and add more confidence in the estimates.

Exercise (Crime Rate)

Let's go back to the example from last week.

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	20517.5999	3277.64269	6.259865	0.00e+00
X	-170.5752	41.57433	-4.102897	9.57e-05

- We want to test whether or not there is a linear association between crime rate and percentage of high school graduates, using a t test with $\alpha = 0.01$. State the hypotheses, decision rule, and conclusion.

Exercise (Crime Rate)

- Estimate β_1 with a 99 % confidence interval. Interpret the interval estimate.