STAC67: Regression Analysis

Lecture 4

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Least Square Estimators

Other criteria. Why square the errors?

We could use least absolute deviations estimates, minimizing

$$Q_1(\beta_0, \beta_1) = \sum_{i=1}^{n} |(y_i - \beta_0 - \beta_1 x_i)|$$

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- **Convenience.** Q is differentiable everywhere, so the normal equations have a closed-form solution. Q_1 is not differentiable at zero and needs an algorithm (linear programming).
- **Optimality.** Under the assumptions of the next slide, least squares is the best estimator of a whole class.

Gauss-Markov Theorem

Theorem 1 (Gauss–Markov)

Consider the simple linear model

$$Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$$

Suppose that the following assumptions (called Gauss-Markov assumptions) concerning the random errors are satisfied:

- Mean zero: $E(\epsilon_i) = 0$
- Constant variance: $Var(\epsilon_i) = \sigma^2$
- Uncorrelated: $Cov(\epsilon_i, \epsilon_j) = 0, \;\; i \neq j$

Then the least square estimators $\hat{\beta}_0$ and $\hat{\beta}_1$ are unbiased and have minimum variance among all unbiased linear estimators (BLUE).

The conclusion holds for **both** estimators separately, and in fact for any linear combination $c_0\beta_0 + c_1\beta_1$: for instance the fitted mean $\hat{\beta}_0 + \hat{\beta}_1 x_0$ is BLUE for $E(Y \mid X = x_0)$. Note that normality is **not** assumed.

Fitted values and Residuals

Definition

Regression equation or fitted regression line

$$\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X,$$

where $\hat{Y}$ is the estimated mean of the response variable at level X of the explanatory.

- For each observation, we can compute the fitted value:

$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i, \quad i = 1, \ldots, n$$

- The vertical distance from the observed y_i to the fitted value is called the residual

$$e_i = Y_i - \hat{Y}_i = Y_i - (\hat{\beta}_0 + \hat{\beta}_1 X_i), \quad i = 1, \ldots, n$$

The residuals can be thought of as predicted values of the unknown errors $\epsilon_1, \ldots, \epsilon_n$.

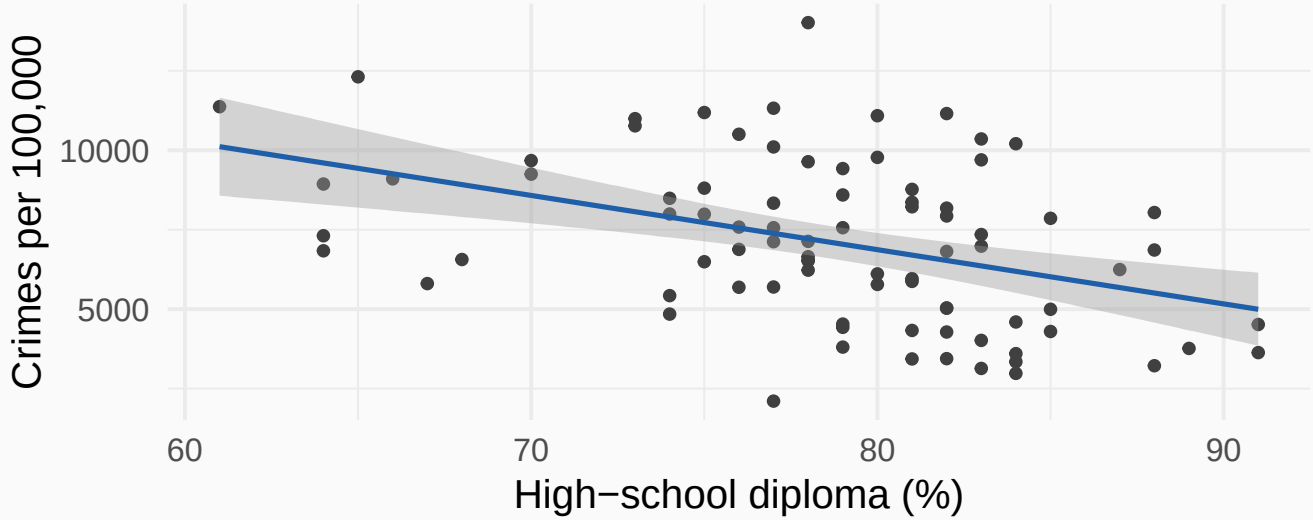
Example: Crime rate

A criminologists studying the relationship between level of education and crime rate in medium-sized U.S. counties collected from a random sample of 84 counties; X is the percentage of individuals having at least a high-school diploma, and Y is the crime rate (crimes reported per 100,000 residents) last year.

```r
library(ggplot2)

Crime <- read.table("CrimeRate.txt")
names(Crime) <- c("Y", "X")

ggplot(Crime, aes(X, Y)) +
  geom_point(size = 0.9, colour = "grey25") +
  geom_smooth(method = "lm", colour = "#1F5FA9", linewidth = 0.6) +
  labs(x = "High-school diploma (%)", y = "Crimes per 100,000") +
  theme_minimal(base_size = 10)
```



Example (Continued)

```
attach(Crime)
fit = lm(Y~X, data=Crime)
coefficients(fit)
```

```
## (Intercept)           X
##  20517.5999   -170.5752
```

```
c(mean(Y), mean(X))
```

```
## [1] 7111.20238   78.59524
```

```
c(sum((Y-mean(Y))*(X- mean(X))), sum((X-mean(X))^2))
```

```
## [1] -547928.119    3212.238
```

```
c(Y[10], X[10])
```

```
## [1] 7932   82
```

```
Crime[10, ]
```

```
##       Y  X
## 10 7932 82
```

```
fit$residuals[10]
```

```
##       10
## 1401.566
```

Exercise

1. Let's compute the estimated regression coefficients by hands:

b_0 :

b_1 :

2. Obtain the point estimates of the following:
   - the difference in the mean crime rate for two counties whose high-school graduation rates differ by one percentage points.
   - the mean crime rate last year in counties with high school graduation percentage $X = 80$.
   - ϵ_{10}

Properties of the fitted regression line

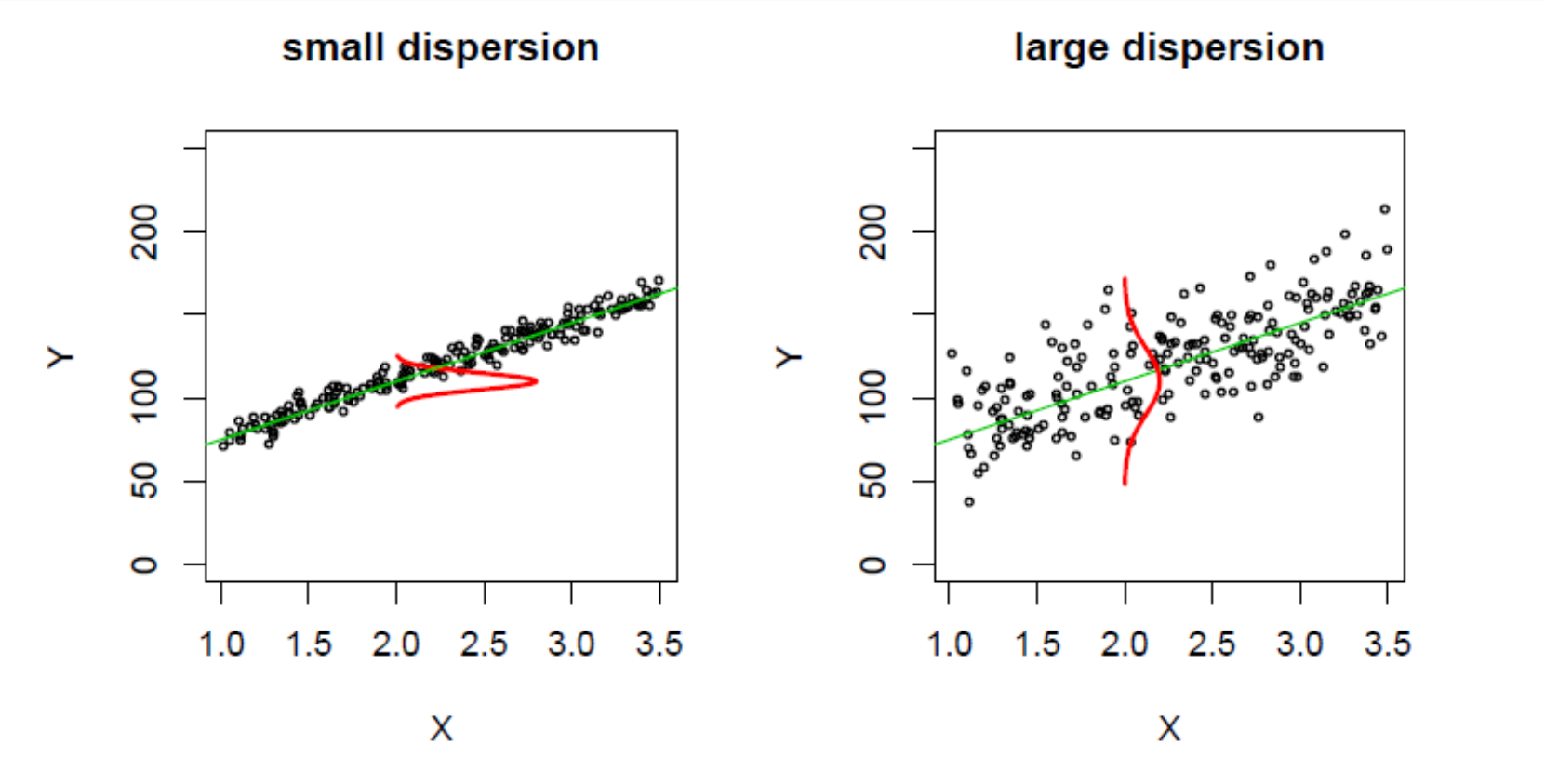
- The least squares line always passes through the point $(\bar{X}, \bar{Y})$.
- The estimated slope $\hat{\beta}_1$ always has the same sign as the sample correlation between X and Y, $\widehat{Cov}(X, Y)$.
- The sum of residuals are 0.
  (i) $\sum e_i = 0$

  (ii) $\sum e_i X_i = 0$

- The sum of squares of the e_i's is called: Residual sum of Squares (RSS) or Sum of Squared Errors (SSE).

The interpretation of σ^2

The variance, σ^2 controls the dispersion of Y around $\beta_0 + \beta_1 X$



Properties of the fitted regression line

- An unbiased estimator of σ^2 is:

- Result:

Properties of Least Squares Estimates

Since $\sum(X_i - \bar{X}) = 0$, the numerator of $\hat{\beta}_1$ simplifies:

$$\sum(X_i - \bar{X})(Y_i - \bar{Y}) = \sum(X_i - \bar{X})Y_i - \bar{Y}\underbrace{\sum(X_i - \bar{X})}_{=0} = \sum(X_i - \bar{X})Y_i$$

so, with $S_{xx} = \sum(X_i - \bar{X})^2$,

$$\hat{\beta}_1 = \frac{\sum(X_i - \bar{X})Y_i}{S_{xx}} = \sum_{i=1}^{n} k_i Y_i, \qquad \text{where } k_i = \frac{X_i - \bar{X}}{S_{xx}}.$$

The constants k_i have several interesting properties:

1. $\sum_{i=1}^{n} k_i = \frac{\sum(X_i - \bar{X})}{S_{xx}} = 0$
2. $\sum_{i=1}^{n} k_i X_i = \frac{\sum(X_i - \bar{X})X_i}{S_{xx}} = \frac{\sum(X_i - \bar{X})(X_i - \bar{X})}{S_{xx}} = \frac{S_{xx}}{S_{xx}} = 1$
3. $\sum_{i=1}^{n} k_i^2 = \frac{\sum(X_i - \bar{X})^2}{S_{xx}^2} = \frac{S_{xx}}{S_{xx}^2} = \frac{1}{S_{xx}}$

In 2, $\sum(X_i - \bar{X})X_i = \sum(X_i - \bar{X})(X_i - \bar{X})$ because $\sum(X_i - \bar{X})\bar{X} = 0$.

Chapter 2: Inferences in Regression

Sampling Distribution of Least Square Estimators

Expected value of LSE

1. $E(\hat{\beta}_1)$ =

2. $E(\hat{\beta}_0)$ =

3. $E(\hat{\sigma}^2)$ =

Sampling Distribution of Least Square Estimators

Variance of least square estimators

1. $Var(\hat{\beta}_1)$ =

2. $Var(\hat{\beta}_0)$ =

Sampling Distribution of Least Square Estimators

Theorem 2

1. $$E(\hat{\beta}_0) = \beta_0, \quad E(\hat{\beta}_1) = \beta_1$$

2. $$Var(\hat{\beta}_0) = \left(\frac{1}{n} + \frac{\bar{X}^2}{S_{xx}}\right)\sigma^2, \quad Var(\hat{\beta}_1) = \frac{\sigma^2}{S_{xx}}$$

The variances in Theorem 2, depends on the unknown value of σ^2. If we replace σ^2 with the unbiased estimator $\hat{\sigma}^2$ and take the square root, we get the standard error of the estimators:

Inference about Regression Parameters

Until now, we did not use any property of the noise ϵ_i beyond the first and second moments.

Let's now focus on the simple Linear Model with **Normal Errors**

$$Y_i = \beta_0 + \beta_1 X_i + \epsilon_i,$$

where

- β_0 and β_1 are parameters
- ϵ_i are i.i.d with normal distribution with mean 0 and variance, σ^2.

$$\epsilon_i \sim N(0, \sigma^2)$$

- Under this model,

$$Y_i | X_i \sim N(\beta_0 + \beta_1 X_i, \sigma^2)$$

Inference about Regression Parameters

- Inference about β_1

- σ is unknown, and must be estimated.

$$T = \frac{\hat{\beta}_1 - \beta_1}{\left(\frac{\hat{\sigma}}{\sqrt{S_{xx}}}\right)} \sim t(n-2)$$

- Review: the Student's t-distribution with k degrees of freedom can be defined as the random variable T

$$T = \frac{Z}{\sqrt{V/k}} \sim t_{(k)},$$

where

Sampling ditribution of $\hat{\beta}_1$

- How to prove

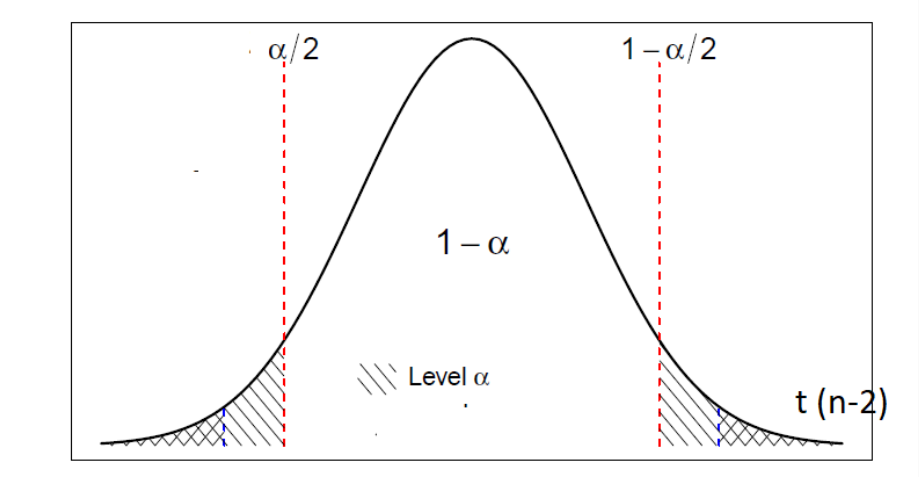
$$\frac{\hat{\beta}_1 - \beta_1}{S_{\hat{\beta}_1}} \sim t(n-2)?$$

Theorem 3: Sampling distributions

$$\frac{\hat{\beta}_0 - \beta_0}{SE(\hat{\beta}_0)} \sim t(n-2), \quad \frac{\hat{\beta}_1 - \beta_1}{SE(\hat{\beta}_1)} \sim t(n-2)$$

Confidence Interval for β_0 and β_1

Since $\frac{\hat{\beta}_j - \beta_j}{SE(\hat{\beta}_j)} \sim t(n-2),\ j = 0, 1$



Confidence Intervals

$$\hat{\beta}_j \pm t(1-\alpha/2; n-2)SE(\hat{\beta}_j), j = 0, 1$$

Why should we care about confidence intervals?

The confidence interval completely captures the information in the data about the parameter.